M2P4 Rings and Fields
Problem Sheet 8.

1. Assume that you have constructed lines AP and BP meeting at an angle θ. Prove that it is possible to construct a line through the origin O which makes an angle θ with the given line OX.

Deduce that if it is possible to construct a regular n-sided polygon somewhere in the plane, then it is possible to construct a regular n-sided polygon inscribed in the circle of radius 1, centred at O.

2. Let $\omega = e^{2\pi i/5}$ and $\alpha = \cos(2\pi/5)$. Prove that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0 \quad \text{text} \quad \omega + \omega^{-1} = 2\alpha.$$

Deduce that $4\alpha^2 + 2\alpha - 1 = 0$ and hence that $\alpha = (-1 + \sqrt{5})/4$.

(1) Why does it follow that it is possible to construct a regular pentagon with ruler and compass?

(2) Prove that the following is a ruler and compass construction of a regular pentagon.

Draw a circle, centre O, with perpendicular radii OX and OY. Let P be the midpoint of OY and let the bisector of the angle XPO meet OX at Q. Construct a perpendicular to OX through Q to meet the circle at A. Then XA is one edge of a regular pentagon inscribed in the circle.

3. It is known that $\cos(2\pi/17) =$

$$\frac{1}{16}\left(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \sqrt{68 + 12\sqrt{17} - 16\sqrt{34 + 2\sqrt{17}} - 2(1 - \sqrt{17})\sqrt{34 - 2\sqrt{17}}}\right)$$

Deduce that it is possible to construct a regular 17-sided polygon with ruler and compass.

4. Let p be an odd prime number. Prove that $1 + x^p + x^{2p} + ... + x^{p(p-1)}$ is the minimal polynomial of $e^{2\pi i/p^2}$ over $\mathbb{Q}$. Deduce that it is impossible to construct a regular polygon with p^2 sides using ruler and compass.