M2P4 Rings and Fields
Answers Sheet 8.

1. Draw a line through O, parallel to AP, to meet the unit circle at A′. Construct OB′ parallel to BP, similarly. Drop a perpendicular from A′ to meet OB′ at C′. Let C lie on OX with OC = OC′. Raise a perpendicular to OX at C to meet the circle at D. Then the angle DOX = the angle APB.

If we can construct a regular n-gon, then we can construct an angle $2\pi/n$. Copy this angle at the origin, as above, and hence obtain a regular n-gon inscribed in the unit circle centred at O.

2. The first two equations are easy. Then we have

$$4\alpha^2 = \omega^2 + \omega^{-2} + 2 = \omega^2 + \omega^3 + 2 = -1 - \omega - \omega^{-1} + 2 = 1 - 2\alpha.$$

Hence $\alpha = (-1 + \sqrt{5})/4$.

(1) α is obtained from $\mathbb{Q}$ by adjoining $\sqrt{5}$. Hence we can construct $\cos(2\pi/5)$ and a regular pentagon.

(2) Let OX $= 1$, the angle OPX $= \theta$ and $t = \tan\frac{1}{2}\theta$. Then $2 = \tan\theta = 2t/(1-t^2)$, so $t = (-1+\sqrt{5})/2$. Hence, OQ $= (-1+\sqrt{5})/4 = \cos(2\pi/5)$, and therefore the angle AOX $= 2\pi/5$.

3. We can construct $\cos(2\pi/17)$. This follows from repeatedly applying the facts that sums, products and quotients of constructible numbers are constructible and that if $a > 0$ is constructible then so is $\sqrt{a}$.

4. Let $\omega = e^{2\pi i/p^2}$ and $f(x) = 1 + x^p + x^{2p} + ... + x^{p(p-1)}$. Then ω is a root of $x^{p^2} - 1 = (x^p - 1)\,f(x)$. But ω is not a root of $x^p - 1$, so it is a root of $f(x)$. Put $x = y + 1$ in $f(x)$. We get

$$f(1+y) = \frac{(1+y)^{p^2} - 1}{(1+y)^p - 1}$$

and, modulo p, this

$$\frac{(1+y^{p^2}) - 1}{(1+y^p) - 1} = y^{p(p-1)}.$$

Therefore, $f(1+y) = y^{p(p-1)} + pg(y)$ where $g(x) \in \mathbb{Z}[x]$. The coefficient of 1 in $1 + (1+y)^p + ... + (1+y)^{p(p-1)}$ is p. Hence $f(1+y)$ is irreducible, by Eisenstein.

Now, $|\mathbb{Q}(\cos(2\pi/p^2) : \mathbb{Q}| = p(p-1)/2$, which is not a power of 2. Therefore, it is impossible to construct a regular p^2-sided polygon with ruler and compass.