m3hsoln5.tex

M3H SOLUTIONS 5. 22.11.2013

Q1 (*Polar equation of a conic*).

In the notation of Q1, the focus-directrix property $PF = e.PL$ is $r = 3(\ell - r\cos\theta)$:

$$r(1 + e\cos\theta) = \ell, \qquad \frac{1}{r} = \frac{1}{\ell}(1 + e\cos\theta).$$

Q2 (*Inverse Square Law of Gravity and conical orbits*).
(i) In polars:

$$x = r\cos\theta, \qquad y = r\sin\theta;$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}, \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta};$$

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\cos\theta(\dot{\theta})^2 - r\sin\theta\ddot{\theta},$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\sin\theta(\dot{\theta})^2 + r\cos\theta\ddot{\theta}.$$

So the components of velocity along and perpendicular to OP are $\dot{x}\cos\theta + \dot{y}\sin\theta = \dot{r}$ and $-\dot{x}\sin\theta + \dot{y}\cos\theta = r\dot{\theta}$ (both obvious). The components of acceleration are $\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - r(\dot{\theta})^2$ along OP, and

$$-\ddot{x}\sin\theta + \ddot{y}\cos\theta = 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

perpendicular to OP.
(ii) If the force is *central* (along OP), then there is no acceleration perpendicular to OP. So $h := r^2\dot{\theta}$ is constant (the angular momentum per unit mass). But if A is the area swept out by the radius vector, $dA = \frac{1}{2}r.rd\theta$: $\dot{A} = \frac{1}{2}r^2\dot{\theta}$. So

$$\dot{A} = \frac{1}{2}h = \textit{constant} :$$

the radius vector sweeps out equal areas in equal times (Kepler's Second Law – equivalent to central forces).
(iii) For a central force: write $u := 1/r$. So by (ii), $\dot{\theta} = h/r^2 = hu^2$,

$$\frac{dr}{dt} = \frac{d}{dt}(1/u) = \frac{d}{du}(1/u)\frac{du}{dt} = -\frac{1}{u^2}.\frac{du}{d\theta}.\frac{d\theta}{dt} = -\frac{1}{u^2}.\frac{du}{d\theta}.hu^2 : \quad \dot{r} = -hdu/d\theta.$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}(-h\frac{du}{d\theta}) = -h\frac{d}{d\theta}(\frac{du}{d\theta}).\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} : \quad \ddot{r} = -h^2u^2d^2u/d\theta^2.$$

So by (i), with a the acceleration a along OP towards O,

$$-a = \ddot{r} - r(\dot{\theta})^2 = -h^2u^2\frac{d^u}{d\theta^2} - \frac{1}{u}.h^2u^4 : \qquad \frac{d^u}{d\theta^2} + u = \frac{a}{h^2u^2}. \qquad (*)$$

Newton's Inverse Square Law of Gravity is $a = c/r^2 = cu^2$ for c constant: $c = a/h^2$ is constant. So by $(*)$, this is

$$\frac{d^2u}{d\theta^2} + u = b \qquad (DE)$$

for some constant b. The differential equation (DE) has general solution $u = b + c_1\cos\theta + c_2\sin\theta$. We may choose the initial line $\theta = 0$ to make $du/d\theta = 0$ there; then $c_2 = 0$, and $u = b + c_1\cos\theta$:

$$1/r = b + c\cos\theta.$$

But the polar equation of a conic of eccentricity e and latus rectum ℓ is

$$r(1 + e\cos\theta) = \ell : \qquad 1/r = \ell^{-1}(1 + e\cos\theta).$$

This identifies the path of a particle moving under the inverse square law as a *conic*. //
(iv) This is the most important single result in Newton's *Principia* (1687), itself the most important book in the history of science. Our orbit round the Sun is closed, so being a conic, it is an *ellipse*. The eccentricity of the ellipse gives us our seasons.

Newton's derivation was geometrical. He avoided calculus (his 'method of fluxions'), thinking it more convincing to solve an old problem by established methods.

The geometrical content, as noted, is Pappus' focus-directrix property of conics (Alexandria, c. 290 AD), used via polar coordinates (cartesian geometry, 17th C.).

The differential equations method above (technically easier for a modern audience) belongs to the 18th C. (the Bernoullis, Euler etc.). Consult any book on Dynamics, e.g.

J. L. SYNGE & B. A. GRIFFITHS, *Principles of mechanics*, McGraw-Hill, 1949.

NHB