m3hsoln6.tex

M3H SOLUTIONS 6. 29.11.2013

Q1 *Viète's infinite product for* π (Francois Viète (1540-1603) in 1593).
(i) The n-gon divides the circle (area π) into n congruent triangles, each of angle $2\pi/n$. Each has area $\frac{1}{2}\sin\pi/n\cos\pi/n = \frac{1}{4}\sin(2\pi/n)$. So

$$A_n = \frac{1}{4}n.\sin(2\pi/n), \qquad A_{2n} = \frac{1}{4}.2n.\sin(\pi/n), \qquad A_n/A_{2n} = \cos(\pi/n).$$

Now $A_4 = 2$ (square of side $\sqrt{2}$), and ($\cos\theta = \sqrt{\frac{1}{2}(1+\cos 2\theta)}$)

$$\cos(\pi/4) = 1/\sqrt{2} = \frac{\sqrt{2}}{2}, \qquad \cos(\pi/8) = \sqrt{\frac{1}{2}(1+\frac{1}{\sqrt{2}})} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\cos(\pi/16) = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}, \ldots.$$

As $A_n \uparrow \pi$, Viète's product follows.

Q2 *Girard's formula of spherical excess* (Albert Girard (1595-1632), *Invention nouvelle en algèbre*, 1629).

On a sphere, a *lune* is the region between two great circles. The ratio of the area of the "A-lune" to that of the sphere is A/π (draw a diagram), and similarly for the B- and C-lunes. If we sum the areas of the three lunes, we cover the area of the sphere, but that of the spherical triangle ΔABC and its antipodal triangle three times (draw a diagram), giving a sum of $S + 4\Delta$ (where S, Δ are the areas of the sphere and triangle). Divide by $S = 4\pi r^2$:

$$\frac{A}{\pi} + \frac{B}{\pi} + \frac{C}{\pi} = 1 + \frac{4\Delta}{4\pi r^2}: \qquad \Delta = r^2(A+B+C-\pi).$$

Q3 (Wallis' product for π).

$$I_n = \int \sin^n x\, dx = -\int \sin^{n-1}x\, d\cos x = -\sin^{n-1}x\cos x + \int \cos x.(n-1)\sin^{n-2}x\cos x\, dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\cos^2 x)\, dx:$$

$$nI_n = -\sin^{n-1}x \cos x + (n-1)I_{n-2}.$$

Passing to the definite integral $J_n := \int_0^{\pi/2} \sin^n x \; dx$ gives

$$J_n = \frac{n-1}{n}.J_{n-2}.$$

So as $\int_0^{\pi/2} dx = \pi/2$, $\int_0^{\pi/2} \sin x \; dx = [-\cos x]_0^{\pi/2} = 1$,

$$J_{2m} = \frac{2m-1}{2m}.\frac{2m-3}{2m-2}.\ldots.\frac{1}{2}.\frac{\pi}{2}, \qquad J_{2m+1} = \frac{2m}{2m+1}.\frac{2m-2}{2m-1}.\ldots.\frac{2}{3}.$$

Dividing,

$$\frac{\pi}{2} = \frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\ldots.\frac{2m}{2m-1}.\frac{2m}{2m+1}.\frac{J_{2m}}{J_{2m+1}}.$$

But as $0 \leq \sin x \leq 1$ in $[0, \pi/2]$, $\sin^{2m+1}x \leq \sin^{2m}x \leq \sin^{2m-1}x$; integrating gives $J_{2m=1} \leq J_{2m} \leq J_{2m-1}$. So

$$1 \leq \frac{J_{2m}}{J_{2m-1}} \leq \frac{J_{2m-1}}{J_{2m+1}} = 1 + \frac{1}{2m} \downarrow 1 : \qquad \frac{J_{2m}}{J_{2m+1}} \to 1 \qquad (m \to \infty).$$

So

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\ldots.\frac{2m}{2m-1}.\frac{2m}{2m+1}.$$

As $2m/(2m+1) \to 1$, this gives

$$\frac{2^2}{3^2}.\frac{4^2}{5^2}.\ldots.\frac{(2m-2)^2}{(2m-1)^2} \to \frac{\pi}{2},$$

which is Wallis' product.

Note.

Take square roots and multiply top and bottom by $2.4.\ldots.(2m-2).2m.\sqrt{2m}$. In the numerator $2^2.4^2.\ldots.(2m-2)^2(2m)^2 = 2^{2m}(m!)^2$, giving

$$\frac{2^{2m}(m!)^2}{(2m)!\sqrt{m}} \to \sqrt{\pi} : \qquad \binom{2m}{m}.\frac{1}{2^{2m}} \sim \frac{1}{\sqrt{m\pi}} \qquad (m \to \infty),$$

useful in Probability Theory.

NHB