

M3H SOLUTIONS 7. 6.12.2013

Q1 (*Euler characteristic; Euler's polyhedron formula*).

To show $F + V = E + 2$, or $V - E + F = 2$: build up the polyhedron face by face, and count $V - E + F$ after each face is added.

1st face. $V = E$ (a polygon has as many vertices as edges), so $V - E + F = 1$.

2nd face. If the face is a p -gon, only $p - 2$ vertices are 'new', and only $p - 1$ edges are 'new'; one face is new. So the $(V-E+F)$ -count increases by $(p - 2) - (p - 1) + 1 = 0$, and stays at 1.

3rd face. If this is a q -gon: $q - 3$ new vertices, $q - 2$ new edges, 1 new face: count changes by $(q - 3) - (q - 2) + 1 = 0$, so stays at 1.

... *Penultimate face.* Similar;y: no change; count stays at 1.

Last face. No new vertices or edges; 1 new face; count up from 1 to 2. //

Q2 (*Duality*).

For projective geometry in the plane, the dual of a statement (involving only incidence, not distance) interchanges the words 'point' and 'line'.

For projective geometry in 3 dimensions, duality interchanges the words point and plane, and leaves the word line unchanged.

So duality sends $(V, E, F) \rightarrow (F, E, V)$, so leaves $V - E + F$ unchanged. So of the Platonic solids, the tetrahedron is self-dual, the cube and octahedron are dual, and the dodecahedron and icosahedron are dual.

Q3 (*Truncated solids*).

If q faces meet at a vertex, and the vertex is truncated ('shaved off'): 1 vertex is lost, q are gained, so V increases by $q - 1$; $V - E + F$ increases by $(q - 1) - q + 1 = 0$: χ is unchanged by truncation (at any vertex, so at all vertices).

For the Platonic solids (p q -gons meet at each vertex): truncation takes

$$V \mapsto V' = qV, \quad E \mapsto E' = E + qV, \quad F \mapsto F' = F + V,$$

$$\chi = V - E + F \mapsto \chi' = V' - E' + F' = qV - E - qV + F + V = V - E + F = \chi = 2 :$$

the Euler characteristic is unchanged by truncation (as we know from Q1).

Note. The Greeks had all 13 Archimedean solids, and the 5 Platonic solids – considerable numerical evidence – and they still missed Euler's formula!

Q4 (*Stirlings's formula: James Stirling (1692-1730) in 1730*).

There are proofs in any decent Analysis book; this one is from
J. C. BURKILL, *A first course in mathematical analysis*, CUP, 1962, Ex.
7(e) Q6.

Consider $d_n := \log(n!) - (n + \frac{1}{2}) \log n + n$.

$$\begin{aligned} d_n - d_{n+1} &= -\log(n+1) - (n + \frac{1}{2}) \log n + (n + \frac{3}{2}) \log(n+1) - 1 \\ &= (n + \frac{1}{2}) \log\left(\frac{n+1}{n}\right) - 1 = \frac{2n+1}{2} \log\left(\frac{1 + (2n+1)^{-1}}{1 - (2n+1)^{-1}}\right) - 1. \end{aligned}$$

But for $|x| < 1$,

$$\begin{aligned} f(x) &:= \frac{1}{2x} \log\left(\frac{1+x}{1-x}\right) - 1 = \frac{1}{2x} \left[\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \dots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \dots\right) \right] - 1 : \\ f(x) &= \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 \dots \end{aligned}$$

In particular, $f(x) \geq 0$, so $d_n - d_{n+1} \geq 0$: $d_n \downarrow$.

Also, for $|x| < 1$, by above

$$f(x) \leq \frac{1}{3}(1 + x^2 + x^4 + \dots) = \frac{x^2}{3(1 - x^2)}.$$

So

$$\begin{aligned} d_n - d_{n+1} &\leq \frac{(2n+1)^{-2}}{3[1 - (2n+1)^{-2}]} = \frac{1}{3[(2n+1)^2 - 1]} = \frac{1}{3(4n^2 + 4n)} = \frac{1}{12n(n+1)} \\ &= \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right) : \quad d_n - \frac{1}{12n} \uparrow. \end{aligned}$$

So d_n is bounded below, and as it decreases, d_n converges – $d_n \downarrow d$, say. So

$$\exp\{d_n\} = \frac{n!}{e^{-n}n^{n+\frac{1}{2}}} \rightarrow A := e^d : \quad n! \sim Ae^{-n}n^{n+\frac{1}{2}} \quad (n \rightarrow \infty).$$

But (Solutions 6) $\binom{2n}{n} \cdot \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{n\pi}} \quad (n \rightarrow \infty)$. Hence (check) $A = \sqrt{2\pi}$, giving Stirling's formula.

Note. The $\sqrt{2\pi}$ in Stirling's formula is the $\sqrt{2\pi}$ in the standard normal density of Probability Theory and Statistics.

NHB