

**M3H SOLUTIONS 2. 29.10.2016**

Q1 (Angle at centre twice angle at circumference).

Let the chord be  $AB$ ,  $C$  be the point on the circumference,  $O$  the centre of the circle. Required  $\angle AOB = 2\angle ACB$ . Let  $\theta := \angle OAC$ ,  $\phi := \angle OBC$ . Triangles  $\triangle AOC$ ,  $\triangle BOC$  are isosceles (two sides are the radius,  $r$  say). So  $\angle OCA = \theta$ ,  $\angle OBA = \phi$ . So  $AB$  subtends  $\angle ACB = \theta + \phi$  at the circumference. In  $\triangle AOC$ ,  $\angle AOC = \pi - 2\theta$  (angle sum is  $\pi$ ), and similarly  $\angle BOC = \pi - 2\phi$ . The three angles are  $O$  sum to  $2\pi$ ; the two just mentioned sum to  $2\pi - 2\theta - 2\phi$ . So  $\angle AOC = 2(\theta + \phi) = 2\angle ACB$ . //

Note that if the chord goes through the centre, the angle at the centre is  $\pi$ , so the angle at the circumference ('angle in a semi-circle') is  $\pi/2$ , and we recover the theorem of Thales.

Q2 (Angles in the same segment).

Both angles subtend the same angle at the centre, so by Q1 they are equal.

Q3 (Opposite angles of a cyclic quadrilateral sum to  $\pi$ ).

If the opposite angles are  $\theta := \angle ABC$ ,  $\phi := \angle ADC$ : by Q1, the arc  $ABC$  subtends angle  $2\theta$  at  $O$ , and arc  $ADC$  subtends  $2\phi$  at  $O$ . But these angles sum to  $2\pi$  (the total angle at  $O$ ). So  $\theta + \phi = \pi$ . //

Q4 (Schläfli symbols and Platonic solids).

(i) As in the star pentagram: as we go round the perimeter of a regular  $p$ -gon, the direction changes by  $2\pi/p$  at each vertex. So the interior angle at each vertex is  $\pi - 2\pi/p = \pi(1 - 2/p)$ . But  $q$  of these can fit together in a polyhedron iff  $q \cdot \pi(1 - 2/p) < 2\pi$ . So the required inequality is

$$q(1 - 2/p) < 2.$$

(ii) Tetrahedron: triangular faces, 3 meet at a vertex:  $\{3, 3\}$ .

Octahedron: triangular faces, 4 meet at a vertex:  $\{3, 4\}$ .

Cube: square faces, 3 to a vertex:  $\{4, 3\}$ .

Dodecahedron: pentagonal faces, 3 at a vertex:  $\{5, 3\}$ .

Icosahedron: triangular faces, 5 at a vertex:  $\{3, 5\}$ .

Q5.

Tetrahedron:  $F = 4, V = 4, E = 6, F + V - E = 2$ .

Octahedron:  $F = 8, V = 6, E = 12, F + V - E = 2$ .

Cube:  $F = 6, V = 8, E = 12, F + V - E = 2$ .

Dodecahedron:  $F = 12, V = 20, E = 30, F + V - E = 2$ .

Icosahedron:  $F = 20, V = 12, E = 30, F + V - E = 2$ .

*Note.*

1. That  $F + V = E + 2$  holds for *all* polyhedra: *Euler's formula* (Week 7, L20). It is result on (combinatorial) *topology* (Weeks 9, 10).

2. There is a sense in which the octahedron and cube are *dual*, the dodecahedron and icosahedron are *dual*, and the tetrahedron is *self-dual*. This involves the ideas of *projective geometry* (Weeks 6 and 8).

NHB