m3hsoln77.tex

M3H SOLUTIONS 7. 4.3.2016

Q1 (*Euler characteristic; Euler's polyhedron formula*).

To show $F+V=E+2$, or $V-E+F=2$: build up the polyhedron face by face, and count $V-E+F$ after each face is added.
1st face. $V=E$ (a polygon has as many vertices as edges), so $V-E+F=1$.
2nd face. If the face is a p-gon, only $p-2$ vertices are 'new', and only $p-1$ edges are 'new'; one face is new. So the (V-E+F)-count increases by $(p-2)-(p-1)+1=0$, and stays at 1.
3rd face. If this is a q-gon: $q-3$ new vertices, $q-2$ new edges, 1 new face: count changes by $(q-3)-(q-2)+1=0$, so stays at 1.
... *Penultimate face.* Similarly: no change; count stays at 1.
Last face. No new vertices or edges; 1 new face; count up from 1 to 2. //

Q2 (*Duality*).

For projective geometry in the plane, the dual of a statement (involving only incidence, not distance) interchanges the words 'point' and 'line'.

For projective geometry in 3 dimensions, duality interchanges the words point and plane, and leaves the word line unchanged.

So duality sends $(V,E,F)\to(F,E,V)$, so leaves $V-E+F$ unchanged. So of the Platonic solids, the tetrahedron is self-dual, the cube and octahedron are dual, and the dodecahedron and icosahedron are dual.

Q3 (*Truncated solids*).

If q faces meet at a vertex, and the vertex is truncated ('shaved off'): 1 vertex is lost, q are gained, so V increases by $q-1$; $V-E+F$ increases by $(q-1)-q+1=0$: χ is unchanged by truncation (at any vertex, so at all vertices).

For the Platonic solids (p q-gons meet at each vertex): truncation takes

$$V\mapsto V'=qV,\qquad E\mapsto E'=E+qV,\qquad F\mapsto F'=F+V,$$

$$\chi=V-F+e\mapsto\chi'=V'-F'+E'=qV-E-qV+F+V=V-E+F=\chi=2:$$

the Euler characteristic is unchanged by truncation (as we know from Q1).
Note. The Greeks had all 13 Archimedean solids, and the 5 Platonic solids – considerable numerical evidence – and they still missed Euler's formula!

Q4 (*Stirlings's formula: James Stirling (1692-1730) in 1730*).

There are proofs in any decent Analysis book; this one is from J. C. BURKILL, *A first course in mathematical analysis*, CUP, 1962, Ex. 7(e) Q6.

Consider $d_n := \log(n!) - (n + \frac{1}{2}) \log n + n$.

$$d_n - d_{n+1} = -\log(n+1) - (n + \frac{1}{2}) \log n + (n + \frac{3}{2}) \log(n+1) - 1$$

$$= (n + \frac{1}{2}) \log\Big(\frac{n+1}{n}\Big) - 1 = \frac{2n+1}{2} \log\Big(\frac{1 + (2n+1)^{-1}}{1 - (2n+1)^{-1}}\Big) - 1.$$

But for $|x| < 1$,

$$f(x) := \frac{1}{2x} \log\Big(\frac{1+x}{1-x}\Big) - 1 = \frac{1}{2x}[(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 ...) - (-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 ...)] - 1 :$$

$$f(x) = \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 ...$$

In particular, $f(x) \geq 0$, so $d_n - d_{n+1} = f(1/(2n+1)) \geq 0$: $d_n \downarrow$.

Also, for $|x| < 1$, by above

$$f(x) \leq \frac{1}{3}(1 + x^2 + x^4 + \ldots) = \frac{x^2}{3(1-x^2)} :$$

$$d_n - d_{n+1} \leq \frac{(2n+1)^{-2}}{3[1 - (2n+1)^{-2}]} = \frac{1}{3[(2n+1)^2 - 1]} = \frac{1}{3(4n^2 + 4n)} = \frac{1}{12n(n+1)}$$

$$= \frac{1}{12}\Big(\frac{1}{n} - \frac{1}{n+1}\Big) : \qquad d_n - \frac{1}{12n} \uparrow .$$

So d_n is bounded below, and as it decreases, d_n converges – $d_n \downarrow d$, say. So

$$\exp\{d_n\} = \frac{n!}{e^{-n} n^{n+\frac{1}{2}}} \to A := e^d : \qquad n! \sim A e^{-n} n^{n+\frac{1}{2}} \quad (n \to \infty).$$

But (Solutions 6)

$$\binom{2n}{n} . \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{n\pi}} \qquad (n \to \infty).$$

Hence (check) $A = \sqrt{2\pi}$, giving Stirling's formula.

Note. The $\sqrt{2\pi}$ in Stirling's formula is the $\sqrt{2\pi}$ in the standard normal density of Probability Theory and Statistics.

NHB