m3hsoln8.tex

M3H SOLUTIONS 8. 11.3.2016

Q1 (*Fundamental Theorem of Arithmetic, FTA: Gauss, Disquisitiones arithmeticae, 1801*).

Proof. Existence. Induction. True for $n = 2$. Assume true for every integer $< n$. If n is not prime (i.e. is composite), it has a non-trivial divisor d $(1 < d < n)$. So $n = cd$ $(1 < c < n)$. So each of c, d is a product of primes, by the inductive hypothesis. So n is too, completing the induction.

Uniqueness. Induction. True for $n = 2$. Assume true for every integer $< n$. If n is prime, the result holds, so assume n is composite. If it has two factorisations

$$n = p_1 \dots p_r = q_1 \dots q_s,$$

to show $r = s$ and each p is some q. As p_1 is prime and divides the product $n = q_1 \dots q_s$, it must divide at least one factor (w.l.o.g., q_1): $p_1 | q_1$. Then $p_1 = q_1$ as both p_1, q_1 are prime. Cancel p_1:

$$n/p_1 = p_2 \dots p_r = q_2 \dots q_s.$$

As n is composite, $1 < n/p_1 < n$. Then the inductive hypothesis tells us that the two factorisations above of n/p_1 agree to within order: $r = s$, and $p_2, \dots, p_r$ are $q_2, \dots, q_r$ in some order, as required. //

Historical Note. We owe Mathematics as a subject to the ancient Greeks. Of the 13 books of Euclid's Elements (EUCLID of Alexandria, c. 300 BC), three (Books VI, IX and X) are on Number Theory. From the ordering of the material in Euclid, it is clear that the Greeks knew that they did not have a proper theory of irrationals (i.e. reals). Although they did not state FTA, it had been assumed that they "knew it really", but did not state it explicitly. This view is contradicted by Salomon BOCHNER (1899-1982) (*Collected Papers*, Vol. 4, AMS, 1992). According to Bochner, the Greeks did *not* know FTA, nor have a notational system adequate even to state it!

L. E. DICKSON (1874-1954) (*History of the Theory of Numbers* Vols 1-3, 1919-23) does not address the question of the Greeks and FTA!

The first clear statement and proof of FTA is in Gauss' thesis (C. F. GAUSS (1777-1855); *Disquisitiones Arithmeticae*, 1798, publ. 1801).

Q2 (*Basel problem: Leonhard Euler (1707-1783) in 1735*).
(i) By Fourier Analysis (Joseph Fourier (1768-1830), *Théorie analytique de la chaleur*, 1822).

Write a_n for the Fourier cosine coefficients of $|x|$ on $[-\pi, \pi]$ ($|.|$ is even, so we do not need sine terms). Then

$$\frac{1}{2}a_0 = \frac{1}{2}.\frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx = \frac{1}{\pi}\int_0^{\pi}xdx = \frac{1}{\pi}[\frac{1}{2}x^2]_0^{\pi} = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos\ nxdx = \frac{2}{\pi}\int_0^{\pi}x\cos nxdx = \frac{2}{n\pi}\int_0^{\pi}xd\sin nx$$

$$= \frac{2[x\sin nx]_0^{\pi}}{n\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin nxdx = \frac{2}{n^2\pi}[\cos nx]_0^{\pi} = \frac{2(\cos n\pi - 1)}{n^2\pi}$$

$$= \frac{2((-1)^n - 1)}{n^2\pi} = -\frac{4}{\pi n^2}$$

if n is odd, 0 if $n \neq 0$ is even. So

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m-1)x}{(2m-1)^2}.$$

Putting $x = 0$ gives $0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{odd} 1/n^2$: $\sum_{odd} = \pi^2/8$. But $\zeta(2) = \sum_1^{\infty} 1/n^2 = \sum_{odd} + \sum_{even} = \sum_{odd} + \frac{1}{4}\zeta(2)$: $\frac{3}{4}\zeta(2) = \pi^2/8$, $\zeta(2) = \pi^2/6$.
(ii) By Complex Analysis (Cauchy, 1820s). See e.g. M2P3, L32 (2011).

The infinite product for sin is

$$\frac{\sin z}{z} = \Pi_1^{\infty}(1 - \frac{z^2}{n^2\pi^2}) \qquad (*)$$

Taking $z = \pi/2$ gives *Wallis' product* of 1656 (Problems 6 Q3 by Real Analysis)

$$\pi^{-1} = \frac{1}{2}\Pi_1^{\infty}(1 - \frac{1}{4n^2}).$$

By $(*)$ and the power series for sin,

$$\frac{\sin z}{z} = \sum_{k=0}^{\infty}\frac{(-)^k z^{2k}}{(2k+1)!} = \Pi_1^{\infty}(1 - \frac{z^2}{n^2\pi^2}).$$

Equate coefficients of z^2:

$$-\frac{1}{3!} = -\frac{1}{\pi^2}.\sum_1^{\infty}\frac{1}{n^2}: \qquad \zeta(2) = \sum_1^{\infty} 1/n^2 = \pi^2/6.$$

NHB