m3h11.tex **Week 11, 23.3.2016 only (not examinable)**

20th C.: Late

The mid-20th C. was within living memory[1]. This makes its history more interesting in some ways, as more immediately relevant to us, and how we got here. On the other hand, this is a problem in other ways. We lack the historical perspective on recent events that we have on the more distant past.

The difficulties we mentioned in Week 10 are still present, but in heightened form: there is too much material; it is too technically hard to lend itself to an overview treatment at this level; no one nowadays can hope to be a well-enough rounded mathematician to be knowledgeable across the board. But with that by way of preamble, here is my view on 1950-2000.

The pre-eminence of the US has continued. One reason for this is that the American mathematical pyramid has a broad base. In the US, *all* mathematics ('math') teaching is done by staff ('faculty') in the Math. Department. Furthermore, in the US from age 18 (as in Scotland from age 17), the first year is general. Very large numbers of people learn calculus; they have to be taught. So US Math Departments are very large compared to their European counterparts; this pushes up the 'height of the research pyramid'.

The period 1945-91 was dominated by the Cold War. The US exploded the first atom bombs – fission bombs – in 1945 [2], and proceeded to develop fusion bombs (hydrogen bombs – H-bombs). The USSR (by a combination of espionage and brilliant science) developed its own H-bombs with incredible speed, bearing in mind the devastation of the USSR in WWII. The Space Age was launched with Sputnik in 1947[3]. But by the first men on the Moon (20.7.1969), US supremacy was re-established.

The fraction of the cohort going to university was tiny in 1900, expanded with the new universities in the 60s, and again with the polytechnics in 1992. This has led to a big expansion in academic job opportunities. But universities have been under sustained financial pressure for thirty years now – understandable for expensive lab science – but mathematics is cheap.

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[1]at least for me, b. 1945

[2](test, Los Alamos, 16.7.1945; weapons, Hiroshima, 6.8.1945; Nagasaki, 9.8.1945)

[3](Sputnik, 4.10.1957; the US reply was late, and didn't work – phutnik, Spätnik in German). The US reacted by pouring vast extra resources into science – to the great benefit of US science and math. Similarly for Yuri Gagarin, the first man in space (12.4.1961)

Nicolas Bourbaki

This is a pseudonym for a group of (mainly French) mathematicians, who from 1935 on sought to systematise mathematics in a rigorous way. Founder members included H. Cartan, Weil, Dieudonné and Chevalley. W:
"*Influence on mathematics in general.* Notations introduced by Bourbaki include the symbol for the empty set and a dangerous bend symbol, and the terms injective, surjective, and bijective. The emphasis on rigour may be seen as a reaction to the work of Henri Poincaré, who stressed the importance of free-flowing mathematical intuition, at a cost of completeness in presentation. The impact of Bourbaki's work initially was great on many active research mathematicians world-wide. ... Bourbaki's direct influence has decreased over time. This is partly because certain concepts which are now important, such as ... of category theory, are not covered in the treatise...

The Bourbaki seminar series founded in post-WWII Paris continues. It is an important source of survey articles, written in a prescribed, careful style. The idea is that the presentation should be on the level of absolute specialists, but for an audience which is not specialized in the particular field."
Bourbaki's *Elements of the history of mathematics* (Springer, 1994) is recommended.

ANALYSIS

The old distinction between Real Analysis and Complex Analysis is still there, but its impact has been lessened. Some aspects of complex analysis, and numbers, are limited to dimension 2. The need to work in higher dimensions has driven the search for new methods. This has been most successfully accomplished by the Chicago school of Analysis, founded by Antoni Zygmund (1900-1992) (who left Poland after the beginning of WWII) and Alberto Calderón (1920-96) and Elias M. Stein (1931-). Zygmund is the author of the standard work *Trigonometric series*, I, II, CUP, 1959.[4] Stein is the author (with R. Shakarchi) of the Princeton Lectures in Analysis, I (Fourier, 2003), II (Functional, 2011), III (Real, 2005), IV (Complex, 2003).
Hardy spaces

An interesting and useful blend of complex and functional analysis is the theory of *Hardy spaces*, which grew out of work of Hardy 1915-41 (Works II.3, III.2, IV.2). They have applications in, e.g., the prediction theory of

[4]Zygmund's brilliant pupil and collaborator Józef Marcinkiewicz (1910-40) was killed in the Katyn Massacre.

stationary stochastic processes (NHB, *Probability Surveys* **9** (2012)).

Functional Analysis

To the books by Banach and by Courant & Hilbert have been added:
N. DUNFORD & J. T. SCHWARTZ, Linear operators, Interscience, I (1958), II (1963), III (1971);
M. REED & B. SIMON, *Methods of modern mathematical physics*, I (1972), II (1975), III (1979), IV (1978).

Banach algebras; Operator theory

Banach algebras are Banach spaces with a multiplication (prototype: composition of bounded operators on a Hilbert space). Their study grew out of von Neumann's work on quantum theory, and led to *von Neumann algebras* and C^*-*algebras*, each the subject of a book by J. Dixmier.

PDEs

The successful application of Schwartz distributions to PDEs can be seen in, e.g., the works of the Swedish analyst Lars Hörmander (1931-2012).

Wavelets

Wavelets are a variant on Fourier analysis, where sines and cosines are replaced by, e.g., a 'mother wavelet' $\Delta(t)$ and 'daughter wavelets $\Delta_n(2^j t - k)$ $(n = 2^j + k$ with $k = 0, 1, \dots, n-1)$ – so Δ_n is 'localised' – supported on $[k/2^j, (k+1)/2^j])$. They were developed by Yves Meyer (1939-), Mallat and Morlet, in response to the needs of detection of underwater oil reserves by sonar. They are perfect for *time-frequency* analysis in Statistics, and are the basis for the FBI's digitisation of its fingerprint database (by R. Coifman, a collaborator of Meyer).

A precursor of wavelets was *holography* (*Dennis Gabor* (1900-79); Imperial College 1948-67; FRS 1956; Nobel Prize 1971), which led to the *laser* in 1960.

NUMBER THEORY

We mention here the discovery by Paul Erdös (1913-1996) and Atle Selberg (1917-2007) of an elementary approach to the Prime Number Theorem. Recall that PNT was conjectured for nearly a century before it was proved in 1896 by methods of Complex analysis. See e.g. Hardy & Wright 22.14-16, G. J. O. Jameson, *The Prime Number Theorem* (2003), Ch. 6, NHB, M3P16.

The most famous result of this period was the proof in 1995 of Fermat's last theorem by (now Sir) Andrew Wiles (1953-).

LOGIC AND FOUNDATIONS

Paul J. Cohen (1934-2007) showed in 1963 that AC is *independent* of the other axioms (because its negation is also consistent with them), introducing the method of *forcing*.

Following Cohen's work, a number of other axioms of set theory have been introduced, most notably Gödel's Axiom of constructibility ("$V = L$"); Martin's Axiom (MA) (1970); Jensen's diamond $\diamondsuit$, Ostaszewski's club $\clubsuit$, etc. See e.g. D. H. Fremlin, *Consequences of Martin's axiom*, CUP, 1984.

The *Continuum Hypothesis (CH)* (Cantor, 1878) is that there is no cardinal between those of $\mathbb{N}$ and $\mathbb{R}$. Modern work on CH involves *large cardinals*, e.g. in the work of W. H. (Hugh) Woodin (1955-).

The use of infinitesimals, not regarded as rigorous since the 19th C., was revived by Abraham Robinson (1918-74) in the early 1960s. See e.g.
H. G. DALES & W. H. WOODIN, *Super-real fields*, OUP, 1996.

For background, see e.g.
Thomas J. JECH, *The Axiom of Choice*, North-Holland, 1973;
Jon BARWISE (ed.), *Handbook of mathematical logic*, North-Holland, 1977. This has sections on: Model theory; Set theory; Recursion theory; Proof theory and constructive mathematics.

ALGEBRA

Group theory dominated much of UK algebra during this period, largely led by Philip Hall (1904-1983) [Obit BMFRS 30 (1984); BLMS 16 (1984), 603-626]. Hall was technically a pupil of the statistician Karl Pearson, but was really influenced by Burnside (whom he never met). See the Mathematics Genealogy Project for details of Hall's pupils.

The most spectacular development has been the classification of finite simple groups (from which a classification of finite groups follows, by the Jordan-Hölder theorem). There are four infinite families (cyclic, symmetric, dihedral and alternating) and 26 sporadic simple groups (5 Mathieu groups, from 1861 and 1873, to the *monster group*). Most but not all of the proof is now published (Gorenstein, Lyons & Solomon; Aschbacher & Smith).

Group theory was used in the attack on the German Enigma machines in WWII (below). Group theory and number theory are used in cryptography, and in the growing and important area of computer security.

ALGEBRAIC GEOMETRY

This is the interplay between the geometry of curves, surfaces etc. and the algebraic equations that describe them. It was extensively studied by the Italian school in the 19th C., and later by Hodge and Pedoe in the 20th

C. The area has the off-putting reputation that one has to know everything already before one can even begin to work on it. See however
Miles REID, *Undergraduate algebraic geometry*, CUP, 1988.

The area was revolutionised in the 1950s and 60s by the work of Alexander Grothendieck (1928-2014). We will not give further detail, but remark that Grothendieck is the originator of K-theory, later developed by Atiyah and others, including Daniel Quillen (1940-2011). "Quillen's most celebrated contribution was his formulation of higher algebraic K-theory in 1972. This new tool, formulated in terms of homotopy theory, proved to be successful in formulating and solving major problems in algebra ...". [5]

ALGEBRAIC TOPOLOGY

S. EILENBERG & N. STEENROD, *Foundations of algebraic topology*, Princeton UP, 1952;
H. CARTAN & S. EILENBERG, *Homological algebra*, Princeton UP, 1956;[6]
S. Mac LANE, *Homology*, Springer, 1963.
J. DIEUDONNÉ, *A history of algebraic and differential topology, 1900-1960*, Birkhäuser, 1989
Charles A. WEIBEL, The development of algebraic K-theory before 1980, homepage, Math., Rutgers, 28p.

Algebraic topology has been a particular strength of UK mathematics, largely due to the influence of Henry (J. H. C.) Whitehead (1904-1960) [BM 7 (1961), 349-362], Max (M. H. A.) Newman (1897-1984) [BM 31 (1985); BLMS 18 (1986), 67-72)] and later M. F. (Sir Michael) Atiyah (1929-) [OM; PRS]. Atiyah's 80th birthday was celebrated at the International Centre for Mathematical Sciences (ICMS), Edinburgh at a 2012 conference 'Geometry and Physics: Atiyah 80': "The workshop is planned to coincide with the 80th birthday on 22nd April 2009 of Sir Michael Atiyah PRS, PRSE, Fields and Abel Medallist, and undoubtedly the most influential British mathematician of the last 50 years. ... Perhaps the most exciting interaction between physics and mathematics at the moment is in Quantum Field Theory and String Theory." Although it goes beyond our time-frame 1950-2000, we include this, partly for its description of Atiyah, partly because it shows how closely linked Geometry and Topology are now to Physics. Such a link is that between the work of the physicist Ed Witten (1951-; IAS, Princeton) and S. K. (Sir Simon) Donaldson (1957-) of Imperial College. Atiyah writes of Don-

[5] Grothendieck gave up maths in 1970, over military funding and the Vietnam war.
[6] Henri Cartan (1904-2008) was the son of Elie Cartan.

aldson' work of 1983: "Donaldson's result implied that there are "exotic" 4-spaces, i.e. 4-dimensional differentiable manifolds which are topologically but not *differentiably* equivalent to the standard Euclidean 4-space $\mathbb{R}^4$. What makes this result so surprising is that $n = 4$ is the only value for which such exotic n-spaces exist. These exotic 4-spaces have the remarkable property that (unlike $\mathbb{R}^4$) they contain compact sets which cannot be contained inside any differentiably embedded 3-sphere!"

The Atiyah-Singer index theorem of 1963 states that for an elliptic differential operator on a compact manifold, the analytical index (related to the dimension of the space of solutions) is equal to the topological index (defined in terms of some topological data). It includes results such as the Riemann-Roch theorem, as special cases, and has applications in physics.

PROBABILITY

The first comprehensive book on stochastic processes based on Measure Theory was the classic by J. L. Doob (1910-2004):

J. L. DOOB, *Stochastic processes*, Wiley, 1953.

Stochastic integration dates back to Wiener and Doob in the 1930s, but the modern era begins with the work of Kiyosi Itô (1915-2008) in 1944. Itô showed how *Brownian motion* $B = (B_t)$ may be used as an integrator in stochastic integrals $\int H_u dB_u$ with suitable random integrands $H = (H_t)$. This was later generalised by Paul-André Meyer (1934-2003), Kunita, Watanabe and many others; the resulting stochastic calculus is very useful both in theory and in applications.

Meyer and the French school created the general theory of stochastic processes. For more detail, see e.g. NHB, SP and PfS, and the Dramatis Personae there.

STATISTICS

Bayesian statistics grew, under the influence of de Finetti, Savage, Lindley and others.

Non-parametric statistics and *empiricals* were able to use the power of modern measure-theoretic probability to avoid parametric assumptions.

Parametric statistics benefited from Riemannian geometry. Through the Fisher information, parametric models could be studied geometrically.

Computer-intensive methods came to dominate large areas of statistics: Markov chain Monte Carlo (MCMC), etc.

Wavelets have become a very flexible and useful tool.

Graphical models have been developed, enabling the efficient representation of complex interactions and dependence structures.

ECONOMICS

J. von NEUMANN & O. MORGENSTERN, *Theory of games and economic behaviour*, Princeton UP, 1953 (1st ed. 1944, 2nd 1947).
This highly influential book was a textbook exposition of Game Theory, largely the creation of von Neumann, and its applications to Economics. Utility functions are emphasised.

Fixed-point theorems came to play an important role in mathematical economics, through their link with economic equilibrium.

Mathematical economics was well developed in the USSR. When optimising under constraints, Lagrange multipliers appeared; often these would have interpretations in terms of 'shadow prices', but sometimes such capitalist language would be avoided.

OPERATIONS RESEARCH

This subject emerged from the logistic requirements of WWII – e.g., transporting vast quantities of military stores (weapons, spares, fuel, food) and troops from a large number of ports on the E. coast of the USA to a large number of destinations in Europe or N. Africa, subject to the risk of sinking by U-boats in the Atlantic.

Linear Programming (LP)

This, and the most important method, the *simplex method*, was introduced by George B. Dantzig (1914-2005) in 1947, alas too late for WWII. In brief: to maximise subject to linear constraints, form the appropriate polygon whose edges are given by the constraints. The maximum will be attained at a *vertex*; the simplex method gives an algorithm for finding *which* vertex.

Dynamic Programming (DP)

Perhaps the other most widely used OR technique is *dynamic programming (DP)*, introduced by Richard Bellman (1920-84) in 1957.

MATHEMATICAL FINANCE

This area dates from the work of Louis Bachelier (1870-1946) in his thesis of 1900, *Théorie de las Spéculation*:
Mark Davis & Alison Etheridge: Louis Bachelier's *Theory of Speculation*: The origins of modern finance, Princeton UP, 2006.
Unfortunately Bachelier's work was undervalued at the time, and then forgotten; it was re-discovered much later.

Finance became a science rather than an art with the 1952 thesis of Harry Markowitz (1927-). Markowitz gave us two key insights:
1. Think of risk and return together, not separately (risk $\leftrightarrow$ variance, return $\leftrightarrow$ mean, hence mean-variance analysis).

2. Diversify: hold a range of different assets, with lots of negative correlation (so that 'what one loses on the swings one gains on the roundabouts').
One then e.g. maximises return for a given risk; the optimal *tangent portfolio* corresponds to a tangent to a convex set. This led on to the capital asset pricing model (CAPM, "cap-emm"; Sharpe, Lintner and Mossin, 1960s).

The modern era of mathematical finance began in 1973 with the *Black-Scholes formula* for pricing options (Fischer Black (1938-95), Myron Scholes (1941-)). They derived this from their *Black-Scholes PDE*, a parabolic PDE and variant on the heat equation. Being applied mathematicians, they knew how to solve the heat equation using the fundamental solution or Green's function; they then transformed back, obtaining their formula – the most famous formula of the late 20th C. Robert Merton (1944-) gave an alternative treatment using probabilistic methods.[7] Martingales and stochastic calculus were later shown to be crucial, as were arbitrage arguments.
CONTROL THEORY

From NHB, SMF Day 9: "State-space models originate in Control Engineering. This field goes back to the governor on a steam engine (James WATT (1736-1819) in 1788): to prevent a locomotive going too fast, the governor (a rotating device mounted on top of the engine) rose under centrifugal force as the speed increased, thus operating a valve to reduce the steam entering the cylinders. This was an early form of *feedback control*.

The Kalman filter (Rudolf KALMAN (1930-) in 1960) was a device for online (or real-time) control, suitable for use with linear systems, quadratic loss and Gaussian errors (LQG) (the term *filter* is used because one 'filters out' the noise from the signal to reveal the best estimate of the state). This appeared just when it was needed, for online control of manned spacecraft during the 60s." Control Theory makes extensive use of the *Hamilton-Jacobi-Bellman (HJB) equations*, a combination of Hamilton-Jacobi dynamics (19th C.) and Bellman's DP (above). See e.g. the books of M. H. A. Davis [8], and A. BAIN & D. CRISAN, *Fundamentals of stochastic filtering*, Springer, 2009.

COMPUTERS

The modern computer emerged from the demands of WWII. At Bletchley Park, teams of crytographers and mathematicians worked (sccessfully) on

[7]Scholes and Merton received the Nobel Prize in 1997, Black having died by then.

[8](Professor at Imperial College, of Control Theory (ElecEng) before he left academia for the City, Mathematical Finance after his return)

decding German military and naval radio messages, encrypted using Enigma machines. The operation became known as the Ultra Secret. It became public knowledge only in the 1970s.

Alan Turing (1912-52)[9] went from his Cambridge work on logic to Bletchley Park, where he headed the mathematical group (I. J. (Jack) Good (1916-2009), a pioneer of both Bayesian statistics and artificial intelligence (AI), was Turing's statistical assistant). After the war, he worked in Manchester on the first stored-programme electronic computer. (These machines used thousands of thermionic valves, filled whole rooms, generated lots of heat, needed constant maintenance, and needed programmes written on punched cards. The microchip depended on later advances in the physics of semiconductors – "Silicon Valley", near Stanford and Palo Alto.)

The demands of theoretical computer science have led to new directions in mathematical logic – e.g., we have met computable numbers and recursive functions above.

The change from computers being large main-frames in specialist establishments to being on every desk and in every home has come in the last twenty years or so, as has e-mail and the Internet.

NUMERICAL ANALYSIS

Until the computer age, to find the value of a special function at some point one used mathematical tables, and *interpolation* to use the surrounding accurate values to get the value one wanted, which was invariably 'missing'. Nowadays, one calls up the appropriate subroutine, on specialist mathematical software (Mathematica, Maple). The Calculus of Finite Differences, originally used for e.g. interpolation, is now used instead for e.g. numerical solution of ODEs, PDEs, SDEs etc.

Finite elements have replaced finite differences for many purposes – for example, in CADCAM (computer-aided design and computer-aided manufacture). The curved surfaces of cars, ships, planes etc. are designed in this way. Splines were originally the flexible strips used by draughtsmen for such tasks; they are now the piecewise polynomial functions used to pass from one subinterval (or finite element) to the next, with the values and those of a many derivatives as possible beng continuous.

The *fast Fourier transform (FFT)* (J. W. Cooley & J. W. Tukey, 1965) has greatly improved the efficiency of numerical Fourier analysis.

[9]Turing was homosexual, and committed suicede following a criminal conviction. A Royal Pardon was issued on 23.12.2013.

QUANTUM THEORY

Quantum electrodynamics (QED) – the theory of the interaction of matter with radiation – was evolved by Sin-Itiro Tomonaga (1906-1979) in 1948, Julian Schwinger (1918-1994) in 1949 and Richard Feynman (1918-1988) in 1949 and 1950. They were awarded the Nobel Prize for this in 1965. QED is regarded as the jewel in the crown of theoretical physics. See e.g. Feynman's book *QED* (1985). Here Feynman emphasises the link with Newton's work on *partial reflection of light at a mirror*. What impressed Feynman was that, although Newton was unable to solve this problem, he realised that there is a deep mystery here: this is a quantum phenomenon, and cannot be solved by classical methods.

Quantum field theory (which includes QED, quantum chromodynamics (QCD) – quarks, ...) contains mathematical difficulties ('cancelling infinities'), treated e.g. by renormalisation-group methods; details omitted.

The Standard Model of particle physics emerged in the 1970s, following the electroweak theory. This unified the electromagnetic force and the weak nuclear force (governing radioactive decay); they have since been unified with the strong nuclear force (that holds the nucleus together – or protons would repel each other).

Murray Gell-Mann (1929-; Nobel Prize 1969) – who introduced and named quarks[10] – used representation theory for the group $SU(3)$ in particle physics (Pauli's work involved $SU(2)$). The Standard Model uses $U(1) \times SU(2) \times SU(3)$. See e.g. A. Pais, *Inward bound*, OUP, 1986, A. Connes, *Noncommutative geometry*, AP. 1994. The search for unification with the fourth fundamental force, gravity, remains elusive.[11]

RELATIVITY and COSMOLOGY

Roger Penrose (1931-) introduced *twistors* in 1967 as a possible path to quantum gravity. See e.g.
R. PENROSE & W. RINDLER, *Spinors and space-time.*
Vol. 1: Two-spinor calculus and relativistic fields, CUP, 1987;
Vol. 2: Spinor and twistor methods in space-time geometry, CUP, 1988;
L. P. HUGHSTON, *Twistors and particles*, LNP 97, Springer, 1979.

[10] The proton, for example, is not an *elementary* particle as once thought, but contains three quarks. That there was structure within the proton was found by scattering experiments.

[11] Recall the recent announcement of the discovery of the Higgs boson at CERN, but this is outside our time-frame.

Stephen Hawking (1942-) combined Quantum Mechanics and General Relativity in his discovery of *Hawking radiation* (from black holes) in 1974. He popularised these previously arcane matters in his best-seller *A brief history of time* (1988).

Background microwave radiation was discovered in 1964 by Penzias & Wilson (Nobel Prize 1978). This gave crucial support to the Big Bang theory of the origin of the Universe, some 13 billion years ago, rather than the Steady State theory. It had been known since Hubble's work on red shifts that the Universe was expanding; the Big Bang explained why.

Current problems in cosmology concern *dark matter* and *dark energy.* Their existence (still mysterious) has been proposed to account for discrepancies between current theory and observation.

The Holy Grail of contemporary Physics is a *grand unified theory (GUT)* of all four of the fundamental forces of Nature. String theory has been proposed here. For an interesting account of quantum gravity, and a sceptical view of string theory, see
Lee Smolin, *Three roads to quantum gravity* (2000); *The trouble with physics* (2006).

The four fundamental forces of nature are mediated by bosons: EM by the photon; the weak nuclear force by W and Z bosons (W^+, W^- – antiparticles of each other – and Z, neutral and its own antiparticle); the strong nuclear force by gluons (John Ellis (1946-) in 1974); gravity by the graviton. The graviton is undetected (and probably undetectable); its signature is currently being sought in *gravitational waves.*

FLUID MECHANICS

Much of late 20th C. work has been on aerodynamics, motivated by the needs of supersonic flight. The equations for supersonic flight are hyperbolic (as with the wave equation – hence the shock waves of the supersonic bang, as with rifle bullets), while those of subsonic flight are elliptic.

A prominent figure here was Sir James Lighthill (1924-98); BM 47 (2001).

GEOPHYSICS; HELIOPHYSICS

Sir William McCrea (1904-1999), BM 53 (2007).

McCrea found in 1928 that the Sun is three-quarters hydrogen and a quarter helium (it had previously been thought of as mostly iron!). The source of the Sun's energy, on which life on Earth depends totally, is now known to be fusion of hydrogen into helium (as with a hydrogen bomb).

Sir Harold Jeffreys (1891-1989), BM 36 (1990).

Jeffreys was primarily a geophysicist, and wrote an influential book *The Earth: Its Origin, History and Physical Constitution*, 1924[12]. He was also a pioneer of Bayesian statistics, and wrote an early book on it, *Theory of probability* (1st ed. 1939, 2nd ed. 1960, 3rd ed. 1983). He also wrote (with his wife) 'Jeffreys and Jeffreys', *Methods of mathematical physics*, CUP, 1946.

Continental drift

The possibility that the bulge in the E. coast of S. America might once have fitted into the W. coast of Africa had been suggested early by Abraham Ortelius in 1596, and later by von Humboldt and others. The theory of Continental Drift was proposed by Alfred Wegener (1880-1930) from 1915 on, and initially opposed (by Jeffreys and others). It was confirmed by paleomagnetic studies in the 1950s. Thanks to global positioning systems (GPS), we can now observe continental drift directly. Thanks to underwater robots and television, we can see the ocean floor opening, plot its contours, etc.

Study of the orientation of magnetism in rocks (e.g. in Africa and S. America) led to new developments in statistics on the circle and sphere. Leading figures here were Fisher and Jeffreys.[13]

CHAOS THEORY

The American meteorologist Edward Lorenz (1917-2008) introduced strange attractors in 1963, and the butterfly effect in 1969. The idea is to study the unpredictability of extremely complicated systems, such as the Earth's weather, which can be predicted well on a small enough time-scale (newspapers carry weather forecasts daily, and weekly) but not for large times (weather is unpredictable in detail a fortnight in advance, for instance).

The practical importance of weather forecasting for ordinary life (travel, etc.) is obvious. The underlying mathematics is the area of *non-linearity*. The journal *Non-linearity* was founded in 1988 by the London Mathematical Society and the Institute of Physics. Non-linearity is much harder than linearity, to which most of our previous mathematics has been devoted. One of the main mathematical challenges of the 21st C. is clearly to develop the mathematics of non-linearity further.

[12] Jeffreys was the first to suggest that the earth's core is liquid – but he was a strong opponent of continental drift!

[13] They were both in Cambridge, though in different departments (Fisher was Professor of Genetics; Jeffreys "taught mathematics, then geophysics and finally became the Plumian Professor of Astronomy"), and did not get on.