m3hw7.tex **Week 7. 24 – 26.2.2016**

18th C.

Background.

The calculus of Newton and Leibniz, plus Newton's *Principia*, triggered the Scientific Revolution. This in turn triggered the Agricultural Revolution and the Industrial Revolution (which both started in Britain). This, together with the growth of the British Empire (whose trade was protected by the Royal Navy) led to the UK becoming a world power, for the first time. In France, the Enlightenment (French: Siècle des Lumières; German: Aufklärung)– a movement of philosophers (Voltaire, Diderot, Rousseau; David Hume, Adam Smith; Immanuel Kant) – led to a new freedom of thinking. The French absolute monarchy could not cope with this, plus defeat at sea by the UK in the 1750s, plus the financial crisis caused by French naval expansion in the 1770s (though this did lead to American independence in 1783). The French Revolution followed in 1789, then the Napoleonic wars. Germany continued a mass of small city states and principalities, under the Holy Roman Empire (till this was abolished by Napoleon in 1804).

N. HAMPSON: *The Enlightenment.* Penguin [Pelican], 1968.

A. HERMAN: *The Scottish Enlightenment: The Scots' invention of the modern world.* Fourth Estate, 2001.

The Bernoulli family

The Bernoullis produced more distinguished mathematicians than any other family in history. Of Netherlands origin, tfe family fled to Basel in Switzerland in 1583 to escape preligious persecution in the Spanish Netherlands. A family tree containing 13 Bernoullis over six generations is in B 20.1. We concentrate on the three most important Bernoullis. Jacques (Jacob, Jacobus), Jean and Daniel (I, in each case).

Jacques Bernoulli (1654-1705), Professor of Mathematics at Basel.

Probability.

Bernoulli was the father of probability theory, with his posthumous book *Ars Conjectandi* (AC) of 1713. Part I of AC reprints Huygens' *De ratiociniis in ludo aleae* of 1657, with commentary. Part II is on permutations and combinations, the binomial theorem, and Bernoulli numbers: the coefficients B_n in

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n x^n / n!$$

useful in many areas (e.g. Analytic Number Theory).

The most important single result in AC is *Bernoulli's theorem.* If in n independent tosses of a biased coin, which falls heads with probability p, tails with prob. $q := 1 - p$ ('Bernoulli trials with parameter p', $B(p)$) we observe S_n heads, the *observed frequency* S_n/n of heads converges as $n \to \infty$ to the *expected frequency* p *in probability*, meaning that

$$\forall \epsilon > 0 \qquad P(|S_n/n - p| > \epsilon) \to 0 \qquad (n \to \infty).$$

This – the Weak Law of Large Numbers (WLLN) for Bernoulli trials – is the first theorem giving a precise form to the folklore idea of the Law of Averages (the general case, and the corresponding SLLN, are 20th C.).

For a full account of AC, see Ch. 15-16 in
Anders HALD, *History of probability and statistics and their applications before 1750*, Wiley, 1990/2003.
Other contributions
Polar coordinates; lemniscate of Bernoulli, $r^2 = a \cos 2\theta$; logarithmic spiral $r = ae^{b\theta}$. Suggested the term 'integral' to Leibniz. Considered continuous compound interest: he knew (AC II)

$$\left(1 + \frac{1}{n}\right) \uparrow c < 3.$$

Jean Bernoulli (1667-1748), Professor of Mathematics at Basel (1705-).
Curves: isochrone; tractix; caustics.
Championed Leibniz against Newton in the priority dispute over calculus.
Taught (M. le marquis) G. F. A. de L'Hospital (1661-1704); L'Hospital's book *Analyse des infiniment petits* (1696), the first textbook on differential calculus, contains 'L'Hospital's rule' (probably due to Bernoulli).
Daniel Bernoulli (1700-1782).

Under Peter the Great (1672-1725), the capital of Russia was moved from Moscow to St. Petersburg, on the Baltic.[1] The St. Petersburg Academy was founded by Catherine I, Peter's widow, in 1725.[2] Daniel Bernoulli was Professor of Mathematics, St. Petersburg, 1724-33; he then moved to Basel. His main contributions were to Probability Theory:

[1] founded 1703; Petrograd 1914; Leningrad 1924; St. Petersburg 1991; capital of Russia 1713-28 and 1732-1918

[2] St. Petersburg Academy of Sciences 1725-1917; Russian Academy of Sciences 1917-25 and 1991 on; USSR Academy of Sciences 1925-91

1. The normal law and the Central Limit Theorem (CLT); tables of the normal law.
2. The maximum-likelihood principle (later attributed to Fisher, 20th C.).
3. The St. Petersburg problem. A single play of the 'St. Petersburg game' is to toss a fair coin till it falls heads, receiving a gain of 2^k (£, say) if this occurs at trial k (which happens with probability 2^{-k}). So the expected gain per play is $\sum_{k=1}^{\infty} 2^k.2^{-k} = \sum_1^{\infty} 1 = \infty$. So LLN ($S_n/n \to E[X]$ in probability) does not apply. It can be shown that

$$S_n/(n \log n / \log 2) \to 1 \qquad (n \to \infty) \qquad \text{in probability.}$$

Moral expectation. What is the 'fair price' for a ticket for one play of the St. Petersburg game? The result above shows that this price should *vary* with n, even though the plays are probabilistically the same. Early workers attempted to handle such problems using a concept of 'moral expectation'. Hardly surprisingly (as one needs 20th C. mathematics, and 'moral expectations' is the wrong approach) such attempts did not succeed.

Abraham de Moivre (1667-1754)
The Doctrine of Chances (DC; 1718; 2nd ed. 1738; 3rd ed. 1756, posth.)

De Moivre was born in France, a Huguenote (= Protestant). He fled to England to escape religious persecution after the revocation of the Edict of Nantes in 1685 by Louis XIV.[3] He was elected FRS in 1697.

In 1733 de Moivre derived the normal curve ("error curve": the term "normal" came later)

$$\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$$

in a 7-page note in Latin *Approximatio ...* (circulated, but not published). An English translation of the *Approximatio* was incorporated in the 2nd and 3rd editions of DC. De Moivre showed that in n Bernoulli trials with parameter p

$$P((S_n/n - p) \in [a, b]) \to \int_a^b \phi(x)dx \qquad (n \to \infty)$$

(recall that $P(S_n = k) = \binom{n}{k} p^k q^{n-k}$, so the LHS is this summed over k with $np + a\sqrt{npq} \le k \le np + b\sqrt{npq}$). This is the special case for Bernoulli trials of

[3]The Edict of Nantes of 1598, under Henry IV, gave Huguenots religious freedom, following the St. Bartholomew's Day Massacre of Protestants in 1572.

the "Law of Errors", or *Central Limit Theorem* (CLT). Note that this refines Bernoulli's theorem by telling us *how fast* S_n/n tends to p: at rate $1/\sqrt{n}$, with a normal limit: in modern terminology,

$$((S_n/n) - p)\sqrt{n}/\sqrt{pq} \sim N(0,1).$$

One can see that this result will need an estimate for the large factorials occurring in $\binom{n}{k} = n!/(k!(n-k)!)$; see Stirling's formula, below. For background and details, see Hald, Ch. 22 (DC), 24 (CLT).
De Moivre's theorem: embryonic form of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(see Euler, below); power series (Phil. Trans.).
James Stirling (1692-1770)
Methodus differentialis (1730): p.135:

$$\log n! = (n + \frac{1}{2})\log n - n + \log\sqrt{2\pi} + \sum_1^\infty B_{2k}/((2k-1)n^{2k-1}),$$

in particular (discarding the series), Stirling's formula,

$$n! \sim \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}} \qquad (n \to \infty).$$

Stirling numbers. These are important in combinatorial theory, and in the Calculus of Finite Differences.
Conics: full treatment by modern coordinate methods.
Colin Maclaurin (1698-1746), Professor of Mathematics at Aberdeen and Edinburgh.
Treatise on Fluxions, 1742: *Maclaurin series* (Taylor series about 0).
Euler-Maclaurin sum formula, connecting sums and integrals via series in Bernoulli numbers. For details, see e.g.
[WW] E. T. WHITTAKER & G. N. WATSON, *Modern analysis*, 4th ed., CUP, 1927/46 [1st ed. 1902, 2nd 1915, 3rd 1920], 7.21,
G. H. HARDY, *Divergent series*, OUP, 1949, Ch. XIII.
Treatise on algebra (1748, posth.): 'Cramer's rule' (below).
Brook Taylor (1685-1731), Secretary to the Royal Society
Methodus incrementorum directa et inversal, 1715: 'Taylor's theorem/series'.
Perspective (books of 1715 and 1719); vanishing points.

Christian Goldbach (1690-1764).

Goldbach was a founding member of the St. Petersburg Academy from 1725, and tutor to Peter II (Tsar from 1728). He is remembered for *Goldbach's conjecture* (letter to Euler of 1742): every even number $n \geq 4$ is a sum of two primes. The weaker *Goldbach ternary conjecture* (every odd number $n > 5$ is a sum of three primes) was solved in 2013 by Harald Helfgott.

Leonhard Euler (1707-1783)

Born in Basel, a pupil of Jean Bernoulli and colleague of Daniel Bernoulli, Euler was the greatest of all Swiss mathematicians, one of the greatest mathematicians of all time, and the most prolific mathematician ever (his Collected Works fill $3\frac{1}{2}$ shelves in the Science Library at UCL). [4]

Euler spent 1727-41 at the St. Petersburg Academy, where he succeeded Daniel Bernoulli as Professor of Mathematics in 1733. He held an appointment at the Berlin Academy of Sciences under Frederick the Great, King of Prussia, from 1741-66, returning to St. Petersburg under Catherine the Great in 1766. Blind in one eye from 1735, Euler lost his sight in 1766, but continued to publish prolifically.

Volume 5.3 (1983) of the journal *Mathematical Intelligencer* is dedicated to Euler (on the bicentennial of his death); on p.6, J. Ewing write:

"Euler left us decisive notation that we still use: the symbol e (1727); the use of $f(x)$ for a function (1734); the use of sin, cos, tan for the trigonometric functions (1748); the notation Δ, Δ^2 for finite differences (1755); the use of $\sum$ for a sum (1755); and the letter i for $\sqrt{-1}$ (1777). He also left us ideas – ideas that were developed extensively in the 19th C. (Example: in 1777 he proved the Cauchy-Riemann equations for the real and imaginary parts of an analytic function, and then used line integrals to compute definite integrals.) He profoundly influenced many areas of mathematics – number theory, special functions, complex functions, finite differences, the calculus of variations, differential equations, differential geometry, topology, mechanics, and fluid dynamics. This is even more remarkable since less than half his published work was *in* mathematics. "

Mechanica (1736): Newtonian dynamics. The number e:

$$e := \sum_{n=0}^{\infty} 1/n!; \qquad e = \lim \Big(1 + \frac{1}{n}\Big)^n.$$

[4] To all maths students who 'prefer calculating to proof': Euler was your patron saint – *the* calculator par excellence.

Introductio in analysin infinitorum (1748). Volume I: Infinite series; infinite products; continued fractions.
Euler products. Euler linked the series

$$\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$$

(the 'Riemann zeta function: Riemann, 19th C.) with the *Euler product*

$$\zeta(s) = \prod_p 1/(1 - p^{-s}),$$

where the product extends over all primes p (NHB, M3P16, II.5). Hence: $\sum 1/n$ diverges implies Euclid's theorem: there are infinitely many primes. Euler showed (the *Basel problem*)

$$\zeta(2) := \sum_1^{\infty} 1/n^2 = \pi^2/6$$

(NHB, M2P3 L31; M3P16 Problems 4 Q3, Problems 6 Q2).
Euler-Maclaurin sum formula: see above, under Maclaurin.
Euler's transformation. This is a technique for accelerating the convergence of a slowly convergent series. It leads to the *Euler summation method*, which may be used to sum divergent series. See e.g. Hardy, *Divergent series*, Ch. VIII, IX.

Euler's manipulations of infinite series were often non-rigorous. What he really lacked was complex-analytic ideas such as *analytic continuation* (see e.g. NHB, M2P3, II.8, Hardy, DS, Ch. I). But he was ahead of his time in complex analysis (19th C.), proving the Cauchy-Riemann equations in 1777.
Euler's totient function (or ϕ-function), $\phi(n)$, the number of integers $k \leq n$ coprime to n, important in number Theory.
Euler's theorem: if y is coprime to n, $y^{\phi(n)} \equiv 1 \pmod{n}$.
As a corollary: if n is a prime p, $\phi(p) = p - 1$.
Hence also Fermat's theorem: if p does not divide y, $y^{p-1} \equiv 1 \pmod{p}$.
Volume II: Coordinate geometry of three dimensions. Cylinders; cones; surfaces of revolution. General theory of curves and surfaces. Quadric surfaces: ellipsoids, hyperboloids of one and two sheets; elliptic and hyperbolic paraboloids. For details, see e.g. Coolidge, *Conics and quadrics*.
Euler angles in spherical polar coordinates: (θ, ϕ) (longitude and colatitude).

Euler line: for a triangle, the circumcentre, orthocentre (point of concurrence of the three perpendiculars) and barycentre (= cetroid: point of concurrence of the three medians) are colinear.
Euler identities: $e^{i\pi} + 1 = 0$;

$$e^{i\theta} = \cos\theta + i\sin\theta; \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta});$$

Euler's constant:

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n \to \gamma \qquad (n \to \infty)$$

($\gamma = 0.5772156649$; it is still not known whether γ is irrational or not).
Institutiones calculi differentialis (1755); *Institutiones calculi integralis* (1768-70, Vol I-III)
Ordinary differential equations (ODEs): Integrating factors; linear equations with constant coefficients; homogeneous and non-homogeneous equations; Euler method for numerical solution of DEs.
Partial differential equations (PDEs): Wave equation. See e.g. NHB, MPC2.
Euler's integral for Beta and Gamma: for the Gamma and Beta functions

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt \qquad (x > 0), \qquad B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1} dt \qquad (p, q > 0),$$

Euler found (see e.g. NHB, PfS L5)

$$B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q).$$

Aside on the Gamma function (NHB, M3P16, I.7): It gives a continuous generalisation of the factorial, $\Gamma(n+1) = n!$. Also $\Gamma(\frac{1}{2}) = \sqrt{2\pi}$, explaining the factor $\sqrt{2\pi}$ in the normal density ϕ and Stirling's formula.
Calculus of Variations (CoV). To maximise (or minimise) an integral

$$I := \int_a^b F(x, y, y')dx \qquad (y' = dy/dx)$$

with respect to variation in the function $y = y(x)$: Euler showed in 1744 that the solution satisfies the '*Euler-Lagrange equation*'

$$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0. \qquad (EL)$$

Example: the Brachistrochrone. A particle slides under gravity along a smooth curve from A to B. Find the curve for which the time taken is a minimum (brachos = short, brachistos = shortest + chronos = time, Greek). The problem was sholved by Bernoulli (Jean and/or Jacques: see B 467) in 1696: the solution is a cycloid. This problem led to the development of the CoV. It may easily be solved by the Euler-Lagrange equations.

We recall the roots of CoV in the work of Huygens; we return to it later (Hamiltonian mechanics, 19th C).

Fluid mechanics

Principes généraux du mouvement des fluides. *Hist. Acad. Berlin*, 1755.
De principiis motus fluidorum. *Novi Comm. Acad. Petrop.* xiv.1, 1759.
Euler introduced the *velocity potential.*

Elasticity

Critical loading of beams and struts. Applications: civil engineering; architecture.

Euler's formula for polyhedra: if the numbers of faces, vertices and edges are F, V, E, then

$$F + V = E + 2.$$

This was proved in a letter to Goldbach in 1750 and published in 1752.

The Königsberg problem.

The city of Königsberg had an island below the confluence of two rivers, and seven bridges. The inhabitants had sought in vain for a walk that went over each bridge exactly once. Euler showed that there is no such walk. His solution may be regarded as a first step in Graph Theory, or in Topology.

Fourier analysis. Early indications of what later became Fourier analysis are to be found in the works of Euler, d'Alembert (below) and Clairaut.

Cauchy-Riemann equations. These were found by Euler (1797, posth.), but named for Cauchy (1814) and Riemann (1851).

Jean d'Alembert (1717-1783)

Encyclopaedia: 28 volumes, in collaboration with Diderot (1713-1784, the philosopher of the Enlightenment). In it, he realised the importance of the *limit* concept in calculus (and analysis), but did not formulate it precisely.

Traité de dynamique (1743).

D'Alembert's principle: internal actions and reactions of a system of rigid bodies in motion are in equilibrium.

PDEs. D'Alembert also studied the *wave equation.*

J. H. Lambert (1728-1777)
Theorie der Parallellinien (MS 1766, publ. 1786, posth.)

Recall that for a triangle with angles A, B, C, $A+B+C-\pi$ is *zero* in the plane (Euclidean geometry), and *positive* on the sphere (by Girard's formula of spherical excess). He realised the possibility of a geometry in which $A+B+C-\pi$ is *negative* (constructed in the 19th C. by Beltrami).

Lambert understood clearly the close link between $A+B+C-\pi$ and Euclid's Parallel Postulate. Here he came closer than anyone else to discovering non-Euclidean geometry (19th C.; Bolyai and Lobachevski).

Adrien-Marie Legendre (1725-1833), Professor at the Ecole Militaire
Mémoires par divers savants, x (1785): *Legendre polynomials*, $P_n(x)$. These are the *orthogonal polynomials* (OPs) on $[-1, 1]$ w.r.t. dx; see e.g. Whittaker & Watson XV. These arise in the solution of *Laplace's equation* (19th C., below) by separation of variables in spherical polar coordinates. See e.g.
G. SZEGÖ, *Orthogonal polynomials*, AMS Colloq. Publ. XXIII, 1939/1959.
Eléments de géométrie (1794): an influential textbook (e.g. in 19th C. USA in translation).
Essai sur la théorie des nombres (1797-8), Vols I, II: the first book(s) devoted entirely to number theory. For background, see e.g.
André WEIL, *Number theory: An approach through history from Hammurapi to Legendre*, Birkhäuser, 1984.[5]
Vol. II contains the *Legendre symbol*, and states (as Euler also did) the *Law of Quadratic Reciprocity* (first proved by Gauss: see e.g. Hardy & Wright, *Theory of Numbers*, 6.12, 13): if

$$\left(\frac{p}{q}\right) := +1 \quad \text{if the congruence } x^2 = p \bmod q \text{ is soluble, } -1 \text{ if not,}$$

then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-)^{\frac{1}{4}(p-1)(q-1)}.$$

That is, the simultaneous congruences

$$x^2 \equiv q \mod p, \quad x^2 \equiv p \mod q$$

[5]Hammurapi (Hammurabi), d.c. 1,750 BC, was the sixth king of Babylon, known for Hammurabi's Code, the first written code of law in recorded history.

are both soluble or both insoluble, unless p, q are both congruent to 3 mod 4, when exactly one is soluble.
Prime Number Theorem (PNT): if $\pi(x)$ is the number of primes $p \leq x$, then

$$\pi(x) \sim x/\log x \qquad (x \to \infty).$$

PNT was conjectured by Legendre (and Gauss), but not proved till 1896.
Least squares
Nouvelles méthodes pour la détermination des orbites des comètes (1805) (supplement 1806, 2nd suppl. 1820).

The orbits of planets and comets, being ellipses and so conics, are determined by two parameters. For flexibility in choice of frame of reference, it may be useful to use p parameters, $\theta_1, \ldots, \theta_p$, where p is small. With exact measurements, these could be determined by p observations, $y_1, \ldots, y_p$ say, but in practice the y_i are polluted by measurement error. However, we may take n readings where n is large (the larger the better – much larger than p). The usual set-up is

$$y_i = \sum_{j=i}^{p} x_{ij}\theta_j + \epsilon_i \qquad (i = 1, \ldots, n),$$

where the x_{ij} are known, the ϵ_i are independent errors of measurement, and the θ_j are to be estimated. How should one estimate the θ_j? Legendre suggested the *method of least squares*: minimise the sum of squares

$$SS := \sum_i (y_i - \sum_j x_{ij}\epsilon_j)^2 = \sum(\text{observed - expected})^2.$$

The p conditions $\partial SS/\partial\theta_j = 0$ give p simultaneous linear equations in p unknowns, the *normal equations* NE; the solutions $\hat{\theta}_j$ give the *least-squares estimators* for the parameters. The method of least squares, in Statistics, is of great practical importance; we return to it in connection with Gauss.
Traité des fonctions elliptiques et des intégrales eulériennes (1825-32), I-III. Eulerian integrals: Gamma and Beta functions, etc. Elliptic integrals (first kind, $F(K, \phi)$: DE for a pendulum; second kind, $E(K, \phi)$: arc-length for an ellipse): for $K^2 < 1$,

$$F(K, \phi) := \int_0^\phi d\theta/\sqrt{1 - K^2\sin^2\theta}, \quad E(K, \phi) := \int_0^\phi \sqrt{1 - K^2\sin^2\theta}d\theta.$$

He is known for the Legendre transformation, which is used to go from the Lagrangian to the Hamiltonian formulation of classical mechanics. See Lagrange (below), Hamilton and Gibbs (19th C.).

Joseph-Louis Lagrange (1736-1813)

Born in Turin, he became Professor of Mathematics at the Royal Artillery School there at 16, going to the Berlin Academy (with Euler and d'Alembert) under Frederick the Great in 1766, and the French Academy in 1787.

Calculus of Variations (CoV). This was Lagrange's earliest (and possibly best) work. In 1755 he wrote to Euler about his work on CoV. Euler generously held up publication of his own work, so that Lagrange's work – which Euler thought superior – should get full credit, and advised Frederick to bring Lagrange to Berlin. The 'Euler-Lagrange equations' date from this time.

Minimal surfaces

A surface which passes through two given (closed, non-intersecting) curves and whose surface area is a minimum is called a *minimal surface* (example: soap bubble held between two rings). Lagrange obtained the DE for minimal surfaces in 1760.

Isoperimetry

The curve enclosing maximal area for a given length is a circle; the surface enclosing maximal volume for given surface area is a sphere, etc.

Libration of the Moon (1764)

Why does the Moon always present the same face to the Earth? Lagrange's solution – an early success in the (unsolved) three-body problem (Sun, Earth, Moon) – won him the Grand Prize of the French Academy.

Satellites of Jupiter (1766): Sun, Jupiter + 4 moons: 6-body problem.

Lagrange's theorem for groups: The order of s subgroup of a finite group divides the order of the group. The group concept came later, with Galois and Abel; Lagrange's work of 1770, in which he conjectured the insolubility of the quintic, was a precursor.

Mécanique Analytique (1788) (MA). If T is the kinetic energy (KE), V is the potential function, the *Lagrangian* L is

$$L := T - V = L(q_1, \ldots, \quad {}_n; \dot{q}_1, \ldots, \dot{q}_n; t)$$

(Lagrange introduced this concept in 1773; *Oeuvres* VI, 335). This leads to *Lagrange's equations*:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_r}\Big) - \frac{\partial L}{\partial q_r} = 0 \quad (r = 1, \ldots, n).$$

For a detailed account of *Lagrangian mechanics*, see e.g.
H. GOLDSTEIN, *Classical mechanics*, Addison-Wesley, 1959, 1.4 and 7.1.
Lagrange multipliers

These concern extrema under constraints. To maximise f under the constraint $g = 0$, introduce a *Lagrange multiplier* λ; maximise $f - \lambda g$ (with no constraint); then find λ from the constraint equation and the unconstrained maxima equation $f' - \lambda g' = 0$.

Principle of Conservation of Energy

From Lagrange's equations, one can obtain

$$\frac{d}{dt}(L - \sum \dot{q}_r \frac{\partial L}{\partial \dot{q}_r}) = 0 : \qquad \sum \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h,$$

where h is a constant of the motion. This is the *energy*, and one has the *Principle of Conservation of Energy*. A special case was known to Galileo. "From this elementary particular case the principle was gradually evolved by Huygens, Newton, John and Daniel Bernoulli and Lagrange." See footnote, p.62, of Whittaker's *Analytical Dynamics (AD)*:
E. T. WHITTAKER, *A treatise on the analytical dynamics of particles and rigid bodies*, 3rd ed., CUP, 1927 (1st ed. 1904, 2nd ed. 1917).
Lagrangian interpolation

Lagrange's interpolation formula (1795) is equivalent to but numerically different from Newton's divided-difference interpolation formula.
Napoleon Bonaparte (1769-1821). [6]

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[6] Born in Corsica, Bonaparte entered the Military Academy at Vienne at 10 and was commissioned into the artillery at 15. His teachers were struck by his intelligence (and phenomenal memory), and his aptitude for mathematics in general and for geometry in particular. This led him into the artillery. It was his handling of the French artillery against the British at the Battle of Toulon (1793) that began one of the two most brilliant military careers of all time (with Alexander the Great – also a student of geometry!)

Napoleon's main contribution to science was that his conquests spread the new metric system, in universal use today. He also influenced the careers of most of the mathematicians of his day, including Laplace and Fourier. Fourier became one of Napoleon's Prefects, Laplace became Minister of the Interior, prompting Napoleon to comment that 'he carried the spirit of the infinitely small into the management of affairs.'

Napoleon also founded the Grandes Ecoles in France (Ecole Normale, Ecole Polytéchnique, etc.), which educate a large proportion of the French elite.
Napoleon proved a non-trivial theorem in geometry. See e.g.
J. E. Wetzel, Converses of Napoleon's theorem. Amer. Math. Monthly 99, (1992), 339-51.