**HISTORY OF MATHEMATICS: EXAMINATION SOLUTIONS, 2013-14.**

**Section A: answer 5 questions out of 10; 10 marks each.**

Q1. *The Fundamental Theorem of Algebra.*
*Carl Friedrich Gauss* (1777-1855)

Gauss' doctoral thesis (in Latin: 'A new proof that every polynomial of one variable can be factored into real factors of the first or second degree') was published in 1799.

Despite its name, this result is a theorem of *analysis*, not of *algebra.* Its proof was less rigorous than Gauss' usual standard: he assumed properties of continuous functions later proved by Bolzano. **[3]**

*Augustin-Louis Cauchy* (1789-1857), Professor at the Ecole Polytéchnique and later the Sorbonne.

Cauchy's *Cours d'analyse* (Ecole Polytéchnique, 1821) contains a proof of the Fundamental Theorem of Algebra: every complex polynomial of degree $n$ has $n$ complex roots (counted according to multiplicity). **[3]**

The modern proof uses *Liouville's theorem* (Joseph Liouville (1809-1882), lectures in 1847 – actually published by Cauchy in 1844): an entire (i.e. holomorphic throughout the complex plane $\mathbb{C}$) bounded function is constant. Thus it took over twenty years before the full power of Cauchy's new subject of Complex Analysis was properly brought to bear on the Fundamental Theorem of Algebra – incidentally, revealing in so doing that the result, being a theorem in Analysis, is a misnomer. **[2]**

*N. H. Abel* (1802-29): Insolubility of the quintic (1829).

Although the quintic has five roots, by the Fundamental Theorem of Algebra above, Abel showed that – in contrast to polynomials of degree up to four, which can be solved, as Cardano showed – quintics are *not soluble by radicals*: there can be *no* formula/algorithm/method for expressing the roots in terms of the coefficients. Similarly for polynomials of higher degree. So there is a fundamental split: polynomial equations are soluble by radicals for degree up to 4, but not for degree 5 or higher. **[2]**

[Seen – lectures]

Q2. *The wave equation.*
*Jean D'Alembert* (1717-1783) in 1746).

By considering the forces acting on a small segment of a string under tension and with small displacements, d'Alembert obtained

$$\partial^2 y/\partial x^2 = \frac{1}{c^2}\partial^2 y/\partial t^2,$$

the one-dimensional *wave equation.* Here $c$ has the dimensions of *velocity* $(L/T)$, and has the interpretation of the velocity of a *wave.* **[2]**

If $f$ is an arbitrary (smooth enough – twice continuously differentiable) function, differentiation shows that $f(x + ct)$ is a solution of the wave equation. Similarly, if $g$ is another arbitrary function, $g(x - ct)$ is also a solution. So by linearity, $f(x + ct) + g(x - ct)$ is also a solution to the wave equation. Think of $f$ as the profile of a wave. Then $f(x + ct)$ represents the wave travelling *left* with velocity $c$. Similarly, $g(x - ct)$ represents a wave with profile $g$ travelling to the *right* with velocity $c$. **[2]**

*Higher dimensions.* In two or three dimensions, the wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \qquad [\mathbf{1}]$$

Useful general methods of solution include *separation of Variables.* **[1]**

*Boundary conditions (BCs).* If the string has length $\ell$, and is fixed at the ends $x = 0$ and $x = \ell$, one gets an infinity of solutions, $\sin k\ell = 0$ for $k\ell = n\pi$, $n$ integer, $k = n\pi/\ell$. By linearity of the wave equation, superpositions of these are solutions. This leads directly to *Fourier series* (*Joseph Fourier* (1768-1830); *Théorie Analytique de la Chaleur* (1822)). **[2]**

*The electromagnetic theory of light* (*James Clerk Maxwell* (1831-1879), *A treatise on electricity and magnetism*, Vol. 1, 2, OUP, 1891).

*Maxwell's Equations.* If $E$, $H$ are the electric intensity and the magnetic field in ES (electrostatic) units, $cE$ in EM (electromagnetic) units, *Maxwell's equations* (in a vacuum) lead to $\nabla^2 E = c^{-2}\partial^2 E/\partial t^2$, $\nabla^2 E = c^{-2}\partial^2 E/\partial t^2$. This is the *wave equation*, for propagation of $E$, $H$ with velocity $c$, the ratio of EM to ES units. This was known experimentally (c. $3 \times 10^{10}$ cm/sec., c. 186,000 miles/sec.) to be (approx.) the *speed of light.* Thus, *electromagnetic forces are propagated with the speed of light.* This suggested to Maxwell that *light waves are electromagnetic.* **[2]**

[Seen – lectures]

Q3. *Mechanics up to Newton.*
*Archimedes* (of Syracuse, c.287-212 BC).

Archimedes constructed pulleys, by which he could move ships in Syracuse harbour single-handed. He was able to use this practically, in defending Syracuse against the Romans in the Second Punic War. He also worked on statics (law of the lever, moments, fulcrum etc.), and hydrostatics (Archimedes' Principle: a floating body displaces its own weight of water). **[2]**

*The Method*. In 1899 J. L. Heiberg found a palimpsest (a parchment which has been re-used) in the Library of the Monastery of the Holy Sepulchre in Jerusalem. Beneath the mediaeval liturgical text, he was able to decipher a previously unknown text by Archimedes, *The Method of Mechanical Theorems for Eratosthenes* (briefly, *The Method*). Here Archimedes describes the method which led him to, e.g., his quadrature of the parabola: an application of his 'principle of the lever' via a balancing argument. **[2]**

*Galileo Galilei* (1564-1642).
*The Two New Sciences* (1638): on mechanics. He showed that
(i) a body falling under gravity does so with constant acceleration;
(ii) the trajectory of a projectile is a parabola. **[2]**

*Sir Isaac Newton* (1642-1727); *Principia*, 1687
*Philosophiae naturalis principia mathematica*, 1687/1713/1726.

Newton's *Principia* is the most famous mathematical book ever published, and rightly so. It triggered the Scientific Revolution, and so helped to usher in the modern world. In the Preface, one finds *Newton's Laws of Motion*:
Law I (inertia): A body continues in its state of rest or uniform motion in a straight line unless force is applied to change it.
Law II: Force = Mass × acceleration; $F = ma$.
Law III: To every action there is an equal and opposite reaction. **[2]**

*Dynamics; Celestial Mechanics*

The great challenge of the new astronomy was to explain Kepler's Laws, which had been arrived at empirically. It was suspected that an inverse square law of attraction was the key by, e.g., Hooke and Halley (Edmond Halley (1656-1742); 2nd Astronomer Royal). In 1684, Halley went to Cambridge, and asked Newton what the orbit of a body was under the inverse square law. Newton replied that it was an ellipse. Asked how he knew, he replied that he had calculated it (long before), but could not find the proof. Halley put the fear of being beaten by others (perhaps Hooke) into Newton's mind. This provoked Newton into writing the *Principia*, published in 1687 at Halley's expense. **[2]** [Seen – lectures]

Q4. *The mathematics of length, area and volume.*
*Greek geometry.*

Geometry was one of the strengths, and main interests, of (ancient) Greek mathematics. They knew a great deal (several of Euclid's books are on geometry), including many highly non-trivial area and volume calculations. Thus they knew integral calculus, without the name (and without differential calculus) – Archimedes' method of exhaustion, etc.

Greek definitions of geometrical entities were sketchy: a point is that with no extent; a line is that with no thickness, etc. **[3]**

*Henri Lebesgue* (1875-1941)

Lebesgue took his PhD in 1902, supervised by Emile Borel (1871-1956). In his pioneering thesis 'Intégrale, longueur, aire' (*Annali di Mat.* **7** (1902), 231-259), Lebesgue introduces the new subject of *measure theory*, and a new integral, now called the *Lebesgue measure*. Despite being harder to set up than the Riemann integral, it is much more general and powerful and much easier to manipulate (e.g., in interchanging limit and integral – *Lebesgue's monotone convergence theorem* and *Lebesgue's dominated convergence theorem*). Lebesgue measure is the mathematics of length, area and volume. It is also the mathematics of gravitational mass, electrostatic charge, and probability; in probability, the total mass is one, and the integral is the expectation. The essence of measure theory is *countable additivity*: for $A_n$ disjoint measurable sets with measures $\mu(A_n)$, the measure of their union is the sum of their measures:

$$\mu(\bigcup_0^\infty A_n) = \sum_0^\infty \mu(A_n). \qquad \textbf{[5]}$$

*Non-measurable sets.*

With the development of modern set theory, beginning with Cantor, foundational questions such as the Zermelo-Fraenkel axioms (ZF), the Axiom of Choice (AC), and their union (ZFC) (the ordinary assumptions of modern mathematics) were studied. It was shown by Vitali that, assuming (AC), there exist *non-measurable sets*. Thus some subsets of the plane are too irregular to have an area, etc. Indeed, there is a sense in which (under (AC)) *most* sets are non-measurable. On the other hand, it is consistent to assume axioms other than (AC), under which *all* sets are measurable. **[2]**

[Seen – lectures.]

Q5. *The abacus and its successors.*

*The Abacus* **[4]**

Herodotus (of Halicarnassus, Asia Minor, c. 484 – c. 425 BC), the Greek historian regarded as the 'father of history', records the use an abacus to solve arithmetical problems.

Early counting was done by means of small stones or pebbles (calculus = pebble, Latin; hence calcaria = limestone; chalk; calcium). Calculations were done by arranging pebbles on a flat surface (abax = flat surface, Greek). Boards divided up into squares, as with chess, were used.

The Greeks excelled in trade, and were adept at the calculations needed for commerce. By contact with Greek culture, Greek use of calculating techniques and devices passed to the Romans.

The upshot of all this is that calculating devices needed for the purposes of trade and administration were widely available, and widely used, throughout the advanced civilisations of the ancient world.

The term 'abacus' today refers to a frame containing parallel wires on which beads can move. Such abaci survived to modern times in China, Japan etc. They were made obsolete for calculation by the widespread availability of pocket calculators from the mid-1970s, and for commercial use by automatic tills with liquid-crystal display (LCD), etc.

*John Napier* (1550-1617) and *logarithms.* **[3]**

*Mirifici logarithmorum canonis descriptio* (1614),

*Mirifici logarithmorum canonis constructio* (1619, posth.)

Napier, a Scottish baron, was led to his discovery of logarithms (logos = ratio + arithmos = number) through being told (by Craig) of the use of prosthaphaeresis (by Tycho Brahe). His logarithms were an early form of modern logarithms to base 10. Logs to base 10 were used in schools (together with slide rules) until the pocket calculator arrived in the early 1970s.

*The computer.* **[3]**

The modern computer emerged from the demands of WWII (e.g., Bletchley Park; Turing; Enigma machines). After the war, Turing went from Bletchley Park, where he headed the mathematical group, to Manchester, where he worked on the first stored-programme electronic computer. The change from thermionic valves to microchips depended on later advances in the physics of semi-conductors, as does LCD etc. The change from mainframe computers to PCs has come in the last twenty years or so, as has e-mail and the Internet. [Seen – lectures]

Q6. *The Platonic solids*

How many solid figures are there whose faces are congruent regular polygons?

Triangular faces. How many faces (equilateral triangles) meet at a vertex? If 3, we have a *tetrahedron*; if 4, an *octahedron*; if 5, an *icosahedron* (2 or less would not give a solid; 6 gives instead a triangular tessellation of the plane, again not a solid; there is no room for 7 or more).

Square faces: 3 gives a cube (4 gives a square tessellation of the plane – 'OS grid').

Pentagonal faces. 3 gives a dodecahedron (4 is impossible, as the angle at the vertex of pentagon is $3\pi/5$ and $4 \times 3\pi/5 > 2\pi$).

Hexagonal faces give a honeycomb tessellation of the plane, and not a solid. So the list above is exhaustive. **[3]**

These 5 'regular polyhedra' are called the *Platonic solids*, after Plato's use of them in the Dialogue of Timaeus. Apparently, according to a scholium to Book XIII of Euclid's Elements, the tetrahedron, cube and dodecahedron were known to the Pythagoreans, while Theaetetus (d. 369 BC) found the octahedron and icosahedron (he probably also proved the theorem above: the list is exhaustive). This is perhaps surprising: it is the dodecahedron and icosahedron that seem the hardest, while the octahedron is merely a pyramid (!) reflected in its base-plane. But archaeological support exists: an Etruscan dodecahedron made c. 500 BC of stone was found in 1885 on Monte Loffa near Padua, and there are Celtic examples. Perhaps also the idea of reflection is not as self-evident as it may seem today.

Unfortunately, Plato viewed the 5 solids as endowed with mystical significance, a throwback to the superstition of the Pythagoreans and the Babylonians before them (earth: cube; air: octahedron; fire: tetrahedron; water: icosahedron; universe: dodecahedron). **[3]**

*Natural occurrence*

*Crystals*: Tetrahedron: sodium sulphantimoniate; cube: common salt; octahedron: chrome alum **[1]**

*Living forms*: skeletons of radiolaria (microscopic sea creatures):
Circogonia: icosahedra; Circorrhegma: dodecahedra. **[1]**

*Duality.*

From the point of view of Projective Geometry: the tetrahedron is self-dual; the cube and octahedron are dual; the dodecahedron and icosahedron are dual. **[2]**

[Seen – lectures]

Q7. *The calculus of variations.*
*Christiaan Huygens* (1629-1695). **[2]**
*Huygens' Principle*: Light travels along paths of shortest time. This idea can be traced from the Greeks (Heron – as paths of shortest distance) through Fermat and Huygens to Euler.
*Leonhard Euler* (1707-1783); born in Basel, a pupil of Jean Bernoulli and colleague of Daniel Bernoulli. **[3]**
*Calculus of Variations* (CoV). To maximise (or minimise) an integral

$$I := \int_a^b F(x, y, y')dx \qquad (y' = dy/dx)$$

with respect to variation in the function $y = y(x)$: Euler showed in 1744 that the solution satisfies the '*Euler-Lagrange equation*'

$$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0. \qquad (EL)$$

*Example: the Brachistrochrone.* A particle slides under gravity along a smooth curve from $A$ to $B$. Find the curve for which the time taken is a minimum (brachos = short, brachistos = shortest + chronos = time, Greek). The problem was solved by Bernoulli (Jean and/or Jacques; also by Newton and Leibniz, *Acta Eruditorum*) in 1696: the solution is a cycloid (the locus of a point on a circle rolling without slipping on a plane – also often used for the arches of bridges). This problem led to the development of the Calculus of Variations. It may easily be solved by the Euler-Lagrange equations.
*Joseph-Louis Lagrange* (1736-1813) (Turin; Berlin Academy;
French Academy). **[3]**
*Calculus of Variations (CoV).* This was Lagrange's earliest (and possibly best) work. In 1755 he wrote to Euler about his work on CoV. Euler generously held up publication of his own work, so that Lagrange's work – which Euler thought superior – should get full credit, and advised Frederick to bring Lagrange to Berlin. The 'Euler-Lagrange equations' date from this time. The name *Calculus of Variations* is due to Euler in 1766
*Sir William Rowan Hamilton* (1805-1865), Professor of Astronomy, Trinity College Dublin, 1827-65. **[2]**
*Hamilton's Principle* (1834): With $L$ the Lagrangian, $\int Ldt$ is an extremum. This includes the *Principle of Least Action*, which can be traced back to Maupertuis (1744).
[Seen – lectures and problems]

Q8. *The emergence of the decimal system.*
*Aryabhata*, author of Aryabhatiya (499 AD) **[2]**

Here we find decimal-place notation: 'from place to place each is ten times the preceding'. The modern decimal numerals 1,2,3,4,5,6,7,8,9 are loosely called Arabic in English, but are called Hindu in Arabic; perhaps 'Hindu-Arabic' would be better. These evolved gradually; the key recognition that by use of place notation the same symbol could be used for three as for thirty, etc., had taken place by 595 AD (Indian source: date 346 in decimal notation), and in Western sources by 662 (Sebokt of Syria).

The zero symbol 0 came later. It had emerged by 876 in India (on an inscription in Gwalior), with the modern 0 for zero. The key components of (i) decimal base, (ii) positional notation, (iii) symbols for 0,1,2,...,9 were thus all in place. It seems that the Hindus did not invent any of them, but they did integrate them into (essentially) their modern form.

*Fibonacci, Leonardo of Pisa* (c.1180-1250) **[2]**

Leonardo of Pisa, son of Bonaccio (hence 'Fibonacci') wrote the *Liber Abaci* (Book of the Abacus) in 1202. This was the most influential European mathematical work before the Renaissance, and was the first such book to stress the value of the (Hindu-)Arabic numerals (Fibonacci had studied in the Muslim world and travelled widely in it).

*Nicholas Chuquet* (fl. c. 1500) **[2]**

*Triparty en la science des nombres*, 1484: the most important European mathematical text since the *Liber Abaci*.

Part I: Hindu-Arabic numerals; addition, subtraction; multiplication; division; Part II: Surds; Part III: Algebra; laws of exponents; solution of equations.

*Simon Stevin* (1548-1620) of Bruges (Brugge), Flemish military engineer. **[2]**

*Die Thiende*, *La disme* ('the tenth'), 1585.

Stevin's elementary book did more than any other to popularise the use of decimal fractions (decimals), and to spread awareness of their computational value and superiority.

*Napoleon Bonaparte* (1769-1821). **[2]**

Napoleon's main contribution to science was that his conquests spread the new metric system, in universal use today.

[Seen – lectures]

Q9. *Projective geometry.*
*Girard Desargues* (1591-1661) and *Projective Geometry.* **[4]**

Desargues' first important book was *La Perspective* (1636). This led him on to his introduction of projective geometry in his *Brouillon projet (d'une atteinte aux evenements des rencontres d'une cone avec un plan)* (1639) *Rough draft (of an attempt to deal with the outcome of a meeting of a cone with a plane)*. As background, recall:
(i) Many results in geometry concern only *incidence properties* (whether lines meet, point lie on a line, etc.); these are preserved under projection.
(ii) Often one has to qualify statements because of exceptional cases involving 'infinity' – e.g., two lines in a plane meet in a point (unless parallel). Recall *perspective* (vanishing point = 'point at infinity').

All this led to Projective Geometry, in which it is incidence properties rather than metrical ones that count. Here one works *projectively*, using *homogeneous coordinates* (in which point in a plane has three coordinates rather than two, determined up to a constant multiple). This powerful tool allows much simplification, but involves a thorough-going change of viewpoint.

The *Brouillon Projet*, though pioneering, had little impact at the time: too far ahead of its time; too badly written; too few copies.

Projective methods in geometry, together with analytic (= coordinate) and synthetic (= classical) methods, complete the main tools needed to treat the geometric problems studied up to that time.

*Conics*. Projective methods allow a simple interpretation of conics as sections of circular cones by planes: the conic are the projections of circles. **[1]**

Projective geometry is of great practical importance: it is the basis of computer graphics, hence of virtual reality etc.

*Blaise Pascal* (1623-1662).
*Essay pour les coniques* (1640) (one page!) Pascal's theorem (on hexagons inscribed in a conic), inspired by Desargues' work. **[1]**
*Charles Jules Brianchon* (1785-1864); *Victor Poncelet* (1788-1867).

Poncelet (MS c. 1812, publ. 1862-4, Works I, II) emphasised *duality* in Projective Geometry: in two dimensions, one may interchange the words 'point' and 'line'; in three dimensions one may interchange 'point' and 'plane', leaving 'line' the same. A prime example of duality is *Pascal's theorem* on hexagons inscribed to conics, and *Brianchon's theorem* on hexagons circumscribed about conics – *discovered* by duality. **[3]**
*The Platonic solids.* The cube and octahedron are dual; the dodecahedron and icosahedron are dual; the tetrahedron is self-dual. **[1]** [Seen – lectures]

Q10. *The mathematics of heat.*
*Joseph Fourier* (1768-1830); Fourier became a Prefect under Napoleon.
*Théorie Analytique de la Chaleur* (1822) (*The analytical theory of heat*).

The propagation of heat in a medium is governed by the *heat equation*: with $k$ the thermal diffusivity, the temperature $u$ satisfies

$$\nabla^2 V := \partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = k^{-1}\partial V/\partial t.$$

This is the prototypical *parabolic* partial differential equation (PDE). **[3]**

The theory of *Fourier series* – representing a function $f(\theta)$ by a trigonometric series $\sum(a_n \cos n\theta + b_n \sin n\theta)$ – is closely connected with the study of heat propagation, and both were developed by Fourier, in his pioneering book and earlier. The continuous analogue of this is the *Fourier integral*,

$$\hat{f}(t) := \int_{-\infty}^{\infty} e^{ixt} f(x)dx.$$

Applying the Fourier transform can transform a PDE into an ordinary differential equation (ODE) – much easier to solve. Example: solution of the heat equation. **[2]**

Just as heat *diffuses* through a conducting medium, so the mathematics of *diffusion* is dominated by the Fourier transform. Example: *diffusions* in Probability Theory – prototype: Brownian motion. This originates in the work of *Robert Brown* in 1828 (observing the movement of pollen particles suspended in water); the mathematical theory of Brownian motion as a stochastic process is due to Norbert Wiener (1894-1964) in 1923. **[2]**
*Rudolf Clausius* (1822-1888) *and Thermodynamics*
*Über die bewegende Kraft der Wärme* (1850) [On the moving force of heat]
This closes with the most famous two-sentence passage in the history of science: Die Energie der Welt ist konstant.
Die Entropie der Welt strebt einem Maximum zu.
[The energy of the world is constant. The entropy of the world strives towards a maximum. (Read 'universe' for 'world' in each here.)]

These are the First and Second Laws of Thermodynamics. The first is the Law of Conservation of Energy. The second is that *entropy increases*. Entropy is a measure of disorder: nature moves towards disorder. Thermodynamics is at the heart of heat transfer, which is a large part of Chemical Engineering. **[3]** [Seen – lectures]

**Section B: answer 2 questions out of 4; 25 marks each**.

Write historical accounts of *two* of the following:

Q1. *The connections between mathematics and astronomy, from ancient to modern times.*
*Egypt*. The early Egyptians learned both the 365 days of the year and the 1/4 day ('leap-year') first-order correction. **[1]**
*The Greeks.*
*Aristarchus* (of Samos, fl. c. 280 BC) **[2]**

By observations made of eclipses, etc., Aristarchus was able to estimate the relative sizes of the earth, sun and moon.

There is scholarly debate on this, but to quote T. L. Heath: '... there is still no reason to doubt the unanimous verdict of antiquity that Aristarchus was the real originator of the Copernican hypothesis'.
*Eratosthenes* (of Cyrene, 276 – 194 BC). **[1]**

His most celebrated work is his measurement of the perimeter of the Earth based on comparison of shadows at midsummer noon down wells at Alexandria and Syrene. His figure of 250,000 stades (c. 24,662 miles) compares very well with the modern figures (the Equator and a polar great circle).
*Hipparchus* (of Nicea, c. 180 – c. 125 BC). **[1]**

Hipparchus stands between the Babylonian roots and the later achievements of Ptolemy. He discovered the precession of the equinoxes, drew up star catalogues, and improved measurements of astronomical constants.
*Ptolemy* (of Alexandria, fl. c. 127 – 150 AD); *Almagest.* **[2]**

The *Megiste Syntaxis* ('greater collection') of Ptolemy (known to the Arabs as al-majisti, hence Almagest), the classic on trigonometry in the ancient world, had an enormous influence on mathematics and astronomy, lasting till the time of Copernicus 14 centuries later.
*The Arabs.* **[2]**

Following the establishment of the House of Wisdom in Baghdad by the Caliph al-Mamun in 829, Arab mathematics and astronomy flourished. Hindu astronomical texts, the Siddhanthas, reached Baghdad even earlier.
*Al-Battani* (c. 850-929): On the motion of stars.
*Al-Fargani*: Elements of astronomy (Latin tr., 12th C.).
*Al-Kashi*, d. c. 1436, of Samarkand: the last great Arab astronomer.

*Nicholas Copernicus* (1473-1543) of Thorn (Niklas Koppernigk of Torun, Poland); *De revolutionibus orbium coelestium*, 1543. **[2]**

This work began the modern period of astronomy by expounding the *heliocentric theory* – that the earth and other planets revolve around the sun.
*Astronomical measurement.* Copernicus measured the distances from the Earth and other planets to the Sun. His figures are close to those of Ptolemy.
*Galileo Galilei* (1564-1642). **[2]**

Galileo invented a telescope, with which he began observations in 1609, observing (i) the Mountains of the Moon; (ii) the four Moons of Jupiter; (iii) the phases of Venus (incompatible with the geocentric system, since this shows that Venus orbits round the Sun). These provided crucial observational support for the Copernican theory.
*The Two Chief Systems* (1632). Written in the form of a dialogue between three characters, this book supported the Copernican heliocentric theory, and brought Galileo into conflict with the Inquisition.
*Tycho Brahe* (1546-1601) spent twenty years observing planetary motion. **[1]**
*Johannes Kepler* (1517-1630). **[2]**

Kepler became Brahe's assistant at the Prague Observatory in 1600, and then spent twenty years analysing Brahe's data.
*Astronomia Nova* (1609): contains Kepler's Laws:
1. Planets move about the Sun in elliptical orbits with the Sun at one focus.
2. A radius vector from Sun to planet sweeps out equal areas in equal times.

Kepler's Third Law (1619): with $P$ the period and $a$ the semi-axis of the orbit, $P^2/a^3$ is constant (for all planets – throughout the Solar System).
*Sir Isaac Newton* (1642-1727); *Principia*, 1687. **[2]**

Here Newton uses his new calculus (fluxions) to prove:
Kepler's Second Law $\Leftrightarrow$ only central forces hold;
Kepler's First Law $\Leftrightarrow$ Inverse Square Law of Gravity;
the planets orbit the Sun in elliptical orbits with the Sun at one focus.
*P. S. de Laplace* (1749-1827); *Mécanique Céleste* I-IV (1799-1825) – synthesised the astronomical knowledge of the time. **[1]**
*A.-M. Legendre* (1725-1833), *Nouvelles méthodes pour déterminer des orbites des comètes* (1805): Least squares. **[1]**
*C. F. Gauss* (1777-1855): Orbit of Ceres, 1801; *Teoria Motus*, 1809 (least squares). **[1]**

*Uranus*, 1781 (Herschel); *Neptune*, 1846 (Galle). **[1]**
*Albert Einstein* (1879-1955), *General Theory of Relativity* (1916). Perihelion of Mercury (1915); bending of light by gravity (1919). **[2]**
*Stephen Hawking* (1942-); *A brief history of time* (1985). **[1]**
[Seen – lectures]

Q2. *The development of the real number system* $\mathbb{R}$.
*The Greeks.*

Beginning with *Pythagoras* (c. 580 – c.500 BC) and his school at Croton, the Greek mathematicians dealt with:
(a) *numbers*: first natural numbers $\mathbb{N}$, then integers $\mathbb{Z}$ and rationals $\mathbb{Q}$ by the arithmetic operations;
(b) *lengths*: of geometric line-segments.

No tension between these two was noticed at first, but – perhaps by *Hippasus* (c. 400 BC) it was observed that such natural *geometric* entities as the length $\sqrt{2}$ of the diagonal of the unit square are *irrational*, and so outside the domain of the number system $\mathbb{Q}$. The same proof [by contradiction] shows that $\sqrt{p}$ is irrational for all primes $p$. **[2]**

*Eudoxus* (of Cnidus, c. 408 – c. 335 BC) introduced his *theory of proportion*: "$a/b = c/d$ iff $\forall\ m, n \in \mathbb{N},\ ma \leq / = / \geq nb \ \Rightarrow mc \leq / = / \geq nd$". This enabled a rigorous treatment of proportionality of numbers, or of similarity of geometric figures, to be given. **[2]**

*Euclid* of Alexandria wrote his *Elements*, Books I-XIII, c. 300 BC. The ordering of the material [esp. Books V, Theory of proportions, and X, Incommensurability and surds] was much affected by the technical problems of being unable to treat incommensurables [or irrationals] rigorously.

The Greeks had problems, because they
(a) did not have an adequate theory of irrationals, but
(b) demanded rigorous proofs. **[2]**

*The Middle ages. Thomas Bradwardine* (c. 1290-1349), Archbishop of Canterbury. **[2]**
*Tractatus de continuo* – regarding the real line as a continuum.
*Nicole Oresme* (1323-1382). **[2]**
*De proportionibus proportionum*: contains laws of exponents, and suggesting *irrational* exponents. The book also contains (Ch. III, Prop. V) a realisation that a 'typical' real number is *irrational*.

*Calculus and analysis: Newton and Euler.* **[5]**
*Issac Newton* (1642-1727), *De analysi* (MS 1669, publ. 1711); *Leonhard Euler* (1707-1783), *Introductio ad analysin infinitorum*, 1748.

Newton systematically handled infinite processes (power series expansions, calculus) avoided by the Greeks. Euler's *Introductio* took this rather loose manipulation with infinite processes further. Various people raised questions about rigour – e.g. *Bishop Berkeley* (1685-1753: *The Analyst*, 1734 – infinitesimals are "the ghosts of departed quantities"), and *Bernhard Bolzano* (1781-1848: *Paradoxien des Unendlichen*, 1851, posth.).

*The construction of the reals* $\mathbb{R}$ *in 1872.*
*Richard Dedekind* (1831-1916), *Stetigkeit und die Irrationalzahlen.* **[4]**

Dedekind's construction of $\mathbb{R}$ centres on the idea of a *Dedekind cut* or *section* (Schnitt). The irrational $\sqrt{2}$ divides the rationals $\mathbb{Q}$ into two classes, those $< \sqrt{2}$ and those $> \sqrt{2}$, either of which serves to identify $\sqrt{2}$. The arithmetic operations can be defined on such cuts, turning the class of cuts into a *field*.
*Georg Cantor* (1845-1918). **[4]**
*Cantor's construction of the reals* (*Math. Annalen* **4** (1972), 123-132 and **21** (1883), 545-591).

Call a sequence $a = (a_n)$ of rationals *fundamental*, or *Cauchy*, if $a_{m+n} - a_n \to 0$ $(m, n \to \infty)$, i.e.

$$\forall \epsilon > 0 \quad \exists N = N(\epsilon) \ s.t. \ n > N, \ m \geq 0 \ \Rightarrow \ |a_{m+n} - a_n| < \epsilon.$$

Call $a = (a_n)$, $b = (b_n)$ *equivalent* if $a - b := (a_n - b_n) \to 0$ (this *is* an equivalence relation – check). Cantor *defines* a real number to be an *equivalence class of Cauchy sequences of rationals.*

Dedekind's construction is specific to $\mathbb{R}$, as it depends on the *total ordering* of the reals. Cantor's construction is by *completion*, and extends to any *metric space*. We quote that any *complete ordered field* is algebraically isomorphic to $\mathbb{R}$, and so may be identified with it. **[2]**
[Seen – lectures]

Q3. *The development of probability theory.*
*Girolamo Cardano* (1501-76). **[2]**

Cardano (c. 1526) wrote a book *De Ludo Aleae* (On Dice Games), published posthumously in 1663. This was the first book written (though not the first book published) on Probability Theory.

*Blaise Pascal (1623-1662), Pierre de Fermat (1601-1665).* **[2]**

On 29 July 1654, Pascal wrote a letter to Fermat on 'de Méré's problem' (or 'paradox'). This problem, a combinatorial one, was suggested by gambling. Its solution is now trite, but this illustrates both the humble mathematical beginnings of probability theory and the spur provided by gambling.

*Christiaan Huygens* (1629-1695). **[2]**
*De ratiociniis in ludo aleae*, 1657 [On reasoning in dice games].

This was the first important book published in probability.

*Jacques Bernoulli* (1654-1705), Professor of Mathematics at Basel. **[4]**

Bernoulli was the father of probability theory, with his posthumous book *Ars Conjectandi* (AC) of 1713. This is the first great book on probability. Part I of AC reprints Huygens' *De ratiociniis in ludo aleae* of 1657, with commentary. Part II is on permutations and combinations, the binomial theorem, and Bernoulli numbers.

The most important single result in AC is *Bernoulli's theorem.* If in $n$ independent tosses of a biased coin, which falls heads with probability $p$, tails with prob. $q := 1 - p$ ('Bernoulli trials with parameter $p$', $B(p)$) we observe $S_n$ heads, the *observed frequency* $S_n/n$ of heads converges as $n \to \infty$ to the *expected frequency* $p$ *in probability*, meaning that

$$\forall \epsilon > 0 \qquad P(|S_n/n - p| > \epsilon) \to 0 \qquad (n \to \infty).$$

This gives a precise form to the folklore idea of the Law of Averages.

*Daniel Bernoulli* (1700-1782). **[2]**

His main contributions were to Probability Theory were: (i) the normal law and the Central Limit Theorem (CLT); (ii) the maximum-likelihood principle; (iii) the 'St. Petersburg problem'.

*Abraham de Moivre* (1667-1754) **[4]**
*The Doctrine of Chances* (DC; 1718; 2nd ed. 1738; 3rd ed. 1756, posth.)

De Moivre was born in France, a Huguenot (= Protestant). He fled to England to escape religious persecution, and was elected FRS in 1697.

In 1733 de Moivre derived the normal curve ("error curve")

$$\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$$

in a 7-page note in Latin *Approximatio* ... (circulated, but not published). An English translation of the *Approximatio* was incorporated in the 2nd and 3rd editions of DC. De Moivre showed that in $n$ Bernoulli trials with parameter $p$

$$P((S_n/n - p)\sqrt{n}/\sqrt{pq} \in [a, b]) \to \int_a^b \phi(x)dx \qquad (n \to \infty)$$

(recall that $P(S_n = k) = \binom{n}{k} p^k q^{n-k}$, so the LHS is this summed over $k$ with $np + a\sqrt{npq} \leq k \leq np + b\sqrt{npq}$). This is the special case for Bernoulli trials of the "Law of Errors", or *Central Limit Theorem* (CLT). Note that this refines Bernoulli's theorem by telling us *how fast* $S_n/n$ tends to $p$: at rate $1/\sqrt{n}$, with a normal limit: in modern terminology,

$$((S_n/n) - p)\sqrt{n}/\sqrt{pq} \sim N(0, 1).$$

One can see that this result will need an estimate for the large factorials occurring in $\binom{n}{k} = n!/(k!(n-k)!)$ (Stirling's formula, 1730).

*Least Squares.* **[5]**

*Adrien-Marie Legendre* (1725-1833), Professor at the Ecole Militaire,
*Nouvelles méthodes pour la détermination des orbites des comètes* (1805) (supplement 1806, 2nd suppl. 1820);
*Carl Friedrich Gauss* (1777-1855),
*Theoria motus corporum coelestium in sectionibus conicus solem ambientem* (1809);
*Pierre-Simon de Laplace* (1749-1827), Professor at the Ecole Normale and the Ecole Polytéchnique,
*Théorie Analytique des Probabilités* (1812) (TAP) (2nd ed. 1814, 3rd 1820).

The *method of least squares* originated in work by Legendre and Gauss [Gauss' delay in publication generated an unpleasant priority dispute with Legendre]. In TAP, Laplace achieved the *Gauss-Laplace synthesis*: the normal law now occurred naturally in both the CLT and in least squares.

*Andrei Nikolaevich Kolmogorov (1903-1987)*; **[4]**
*Grundbegriffe der Wahrscheinlichkeitsrechnung*, 1933.

Here Kolmogorov inauguated the modern [= measure-theoretic] era in Probability Theory by harnessing the Measure Theory of *Henri Lebesgue* (1975-1941: thesis of 1903), following earlier work of *Emile Borel* (1871-1956) and *Maurice Fréchet* (1878-1973).
[Seen – lectures]

Q4. *The development of linear algebra.*
*Determinants.* **[5]**

Determinants (which in modern language technically belong to Multilinear Algebra) may be traced to the work of Leibniz (1646-1716) of 1693 (unpublished till 1880), *Maclaurin* (1698-1746, *Treatise on Algebra*, 1748, posth.), *Cramer* (1704-1752: *Cramer's rule* for solution of linear equations, 1752) and *Lagrange* (1736-1813; geometric work of 1775).

The subject came of age with the 84-page paper of 1812 by *Cauchy* (1789-1857); the term *determinant* comes from the *Disquisitiones arithmeticae* (1801) by *Gauss* (1777-1855).
*Matrices.* **[5]**

Logically, matrices precede determinants, but historically the order was reversed.
*Arthur Cayley* (1821-1895);
*J. J. Sylvester* (1814-1897).

The term 'matrix' was introduced in 1850 by Sylvester, and the theory was developed by Cayley in a series of papers in the 1850s, particularly 1858. This is the source – with *Hamilton*'s lectures of 1853 on quaternions – of the *Cayley-Hamilton theorem*: a matrix satisfies fits own characteristic equation.

Other sources from this time include Sylvester's Law of Inertia (1852) and Law of Nullity (1884), and the introduction of *Hermitian* matrices [Hermite, 1855] and *orthogonal* matrices [Hermite, 1854; Frobenius, 1878].

*Hermann Grassmann* (1809-1877); *Ausdehnungslehre* (1844) [Theory of Extensions]. **[4]**

Grassmann's work, which was geometrically motivated, was a major source from which *linear algebra* – the theory of vector spaces, linear transformations between them, and the matrices representing these, etc. – developed. Grassmann's work is also the source of *Grassmann algebras* – important in modern multilinear and tensor algebra.

Another source was the work of the English geometer *W. K. Clifford* (1845-1879).
*Giuseppe Peano* (1858-1932, *Calcolo geometrico secondo l'Ausdehnungslehre de Grassmann, preceduto dalle operazioni della logica deduttiva*, Torino, 1888. **[2]**

Here Peano, explicitly building on Grassmann's work, gives the axiomatic definition of a vector space (whether of finite dimension or not) over the reals, and of linear transformations between them, all in modern notation.

*Quaternions.* **[2]**

*Sir William (Rowan) Hamilton* (1806-1865) discovered *quaternions* in 1843, and published his *Lectures on Quaternions* in 1853. Quaternions became widely used, especially in applied mathematics, e.g. Thomson and Tait, *Treatise on Natural Philosphy*, 1867.

*Vectors.* **[2]**

The American physicist *Josiah Willard Gibbs* (1839-1903) (*Elements of Vector Analysis*, 1881) saw that the sub-theory of quaternions giving ordinary 3-dimensional *vectors* is adequate for most purposes in applied mathematics. The standard machinery of *vector algebra* (vector products, scalar products, triple vector products etc.) and *vector calculus* (grad, div, curl, $\Delta^2$; theorems of Stokes, Gauss and Green) developed so successfully as to have generally superceded quaternions by the end of the 19th C.

*The 20th C.* **[5]**

Linear Algebra now forms one wing, with Abstract Algebra (groups, rings, fields etc.) of *Modern Algebra*, a core part of the undergraduate curriculum. This subject took shape with
B. L. van der Waerden, *Moderne Algebra*, Vol. I (1930), II (1931).

Van der Waerden (1903-1996, PhD 1925, H. de Vries) was a Dutch mathematician much influenced by Emmy Noether (1882-1935, PhD 1907, P. Gordan). Noether and van der Waerden are the main originators of modern algebra, together with their Göttingen colleague David Hilbert (1862-1943).

Van der Waerden's book was in German, and used Fraktur for the equations, as did its eventual English translation. For such reasons, and through passage of time, it has been supplanted as a text by later books, e.g.
Garrett Birkhoff and Saunders MacLane, *A survey of modern algebra*, 1941/1953;
P. M. Cohn, *Algebra*, Vol. 1 (1974/82), 2 (1977/89), 3 (1991).

In the computer age, Numerical Linear Algebra has become a subject in its own right. Classics include Wilkinson's *The algebraic eigenvalue problem* and Golub & van Loan's *Matrix computation*.

Multivariate Analysis in Statistics makes heavy use of Linear Algebra and Matrix Theory. In particular, *singular values decomposition (SVD)*, not found in Algebra texts a generation ago, is now widely used in Statistics (approximation by rank-one matrices) and Numerical Analysis (because of its good numerical stability). N. H. Bingham