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M3H HISTORY OF MATHEMATICS: SOLUTIONS 0

Recall the ten questions posed in Week 0, as ”Problems 0”:

Q1. *Why do dust particles dance in sunbeams?*

This phenomenon was first observed in antiquity [Week 4]:
"*Lucretius Carus* (c. 99 – c. 55 BC); *De rerum natura* [On the nature of things]. Lucretius' poem contains the first documented reference to (what became known in the 19th C. as) *Brownian motion*. Lucretius observed dust particles dancing in sunbeams. [I remember seeing this myself, aged c. 4, and asking my mother why they did this. Please now ask yourselves this question.]"

There are three scales here: *microscopic* (the atoms and molecules of gas in the air), *macroscopic* (us – the observer), and *mesoscopic* – the dust particles. These are *vast* compared to atoms but *tiny* compared to us. This 'triple scale' enables the dust to 'bridge the gap', allowing us to see a molecular effect with the naked eye.

This phenomenon also occurs in liquids. In 1828 the Scottish botanist Robert BROWN (1773-1858) observed under a microscope pollen particles dancing in suspension in water.

The atomic/molecular explanation above is a consequence of Statistical Mechanics [Maxwell, Boltzmann, Gibbs, 19th C., Week 9]. Recall that the *existence* of atoms was disputed until the early 20th C. (the dispute cost Boltzmann his life).

Q2. *Why are there* **two** *tides a day*?

The Earth-Moon dynamical system is driven by Newton's Inverse-Square Law of Gravity. On Earth, the solid earth is fixed, but the sea, being liquid, is not.

Newton showed [*Principia*, 1687, Week 6] that a rigid body behaves like a point mass concentrated at the centre of mass (this needs integral calculus, which Newton developed in the *Principia*).

Imagine a pair of young lovers on a beach, with a full Moon overhead. They see a high tide, for an obvious reason: the Moon overhead *pulls* the sea in front of them *upwards*.

Now imagine a young couple on the far side of the Earth (the Antipodes), again on a beach. The Moon is directly underneath them, so invisible through the Earth. But again they see a high tide, this time for a non-obvious reason. The Earth behaves like a point mass at its centre. This is nearer to the Moon than the beach (by the radius of the Earth), so it feels the Moon's gravity more than the sea in front of the beach. The lovers see a high tide because the sea-bed in front of them is *sucked downwards*.

This phenomenon had been a standing puzzle for long before Newton – who was the first person to understand it. This alone partly explained the enormous impact of the *Principia*. Note that it can be explained in words – no mathematics needed – once one has the concepts in place.

Q3. *Why does the Moon always show the same face to the Earth, rather than revolve on its axis as the Earth does?*

This results from phenomena known as *tidal acceleraration/deceleration* and *tidal locking*.

The Moon is much smaller than the Earth. So the Earth's gravitational field distorts the shape of the Moon, which is not round but pear-shaped, with the point pointing towards the Earth. This distortion built up gradually over time. Although the Moon originally had an atmosphere, and did rotate on its axis, the frictional drag caused by the atmosphere on the bulge gradually caused a loss of the angular momentum necessary to maintain the rotation. Now that this has ceased, the 'extra' gravitational pull of the Earth on the bulge keeps the bulge pointing towards the Earth; this serves to stabilise the Earth-Moon as a dynamical system. The far side of the Moon was not observed till 1959, from a Soviet spacecraft. [There is a slight wobble, so we can see more that 50% of the Moon's surface over time – *libration*.]

Such tidal locking is quite common between planets and their satellites. See Wikipedia for other examples in the Solar System.

None of this could have been studied before Newton's *Principia* of 1687. This sort of question goes back to E. Halley in 1695 (recall Newton wrote the *Principia* because of Halley! – Week 6). The main advance was by Laplace in 1786 [*Mécanique Céleste*, Vol. I, 1799; Week 8], followed by J. C. Adams in 1853.

Q4. *Why is the sky blue?*
Lord Rayleigh (J. W. Strutt) (1842-1919), 7th President LMS (1876-8), PRS (1905-8), Nobel Prize 1904.
The theory of sound, vol. I, II, 1877 (Dover, 1945)

Rayleigh's work on *scattering* [Week 9] explained why the sky is blue (the earth's atmosphere scatters sunlight – the sky is black in space). For, the scattering is done by photons in the Sun's light interacting with the electric forces within the atoms of the dust particles. Recall that (Maxwell – Week 9) light is electromagnetic. The scattering is done differentially according to wavelengths. The red end of the visible spectrum is scattered more than the blue end, which thus survives to strike the eye.

Q5. *Why do we sweat?*

We sweat because we are warm-blooded. Our bodies are thus very sensitive to temperature. We sweat to cool down – just as we shiver to warm up, to get our muscles moving.

When we sweat, the sweat drops have surface tension, as with any water surface. The water molecules are in thermal oscillation, or bombardment. They have a temperature distribution – the *Maxwell-Boltzmann distribution* [Week 9] of Statistical Mechanics. It is only the more energetic (fast-moving, hotter) molecules that have enough momentum to break out of the drop against the force of the surface tension. Thus evaporation of sweat cools one by draining the sweat differentially of its 'hotter' molecules, so we lose heat and cool down.

Note that this does not work if the air is *humid*, as then it is harder for evaporation to take place. We still lose water through sweat (so risk becoming dehydrated), but do not get the benefit of the cooling (so our core temperature will rise, if we are taking strenuous exercise). This is why humid conditions are feared by endurance athletes such as marathon runners: this can lead to heat-stroke (I should know – I've been there!).

Q6. *Why does water boil? Why does water freeze?*

Boiling. As in Q5: at a high enough temperature, most of the water molecules are energetic enough to break through the water surface.

Phase transitions such as boiling of water involve an *energy-entropy competition.* From Kelvin [Week 9], a system moves towards equilibrium by minimising its energy – a process known as *relaxation.* From Clausius' Second Law of Thermodynamics [Week 9], entropy (a measure of disorder) increases. The liquid state is more ordered than the gaseous, so has lower entropy; but as its molecules move more slowly, has lower energy too. What the boiling point is – where the balance is struck between the tendency to decrease energy (liquid) and increase entropy (gas) depends on the specifics of the chemistry of the substance – water in this case.

This insight followed the study by Onsager of the two-dimensional Ising model for a ferromagnet [Week 10].

The Law of Conservation of Energy (First Law of Thermodynamics) still applies, of course. A phase transition (at least a 'first-order' one, of the type described here) involves a release or absorption of heat. When steam condenses, it releases *latent heat* (latent: hidden; lateo, latere: to hide, Latin). This explains why steam burns are often worse than scalding by hot water. Study of the thermodynamics of phase changes is important in Physical Chemistry (or Chemical Physics) and Chemical Engineering; it involves the *specific heats* (heat capacity) of the substances in question.

Freezing. This is again a phase transition, caused by *energy-entropy competition.* Again, the solid state has lower entropy (being more ordered), and lower energy (its molecules do not move, as in a crystal, or move slowly, as in an amorphous solid). Again what the freezing point is – where the balance is struck between the tendency to decrease energy (solid) and increase entropy (liquid) depends on the specifics of the substance – water here.

When water freezes – to ice, or snow – it does so into crystals, which are ordered. It is easier for the slower-moving (so colder) water molecules to attach themselves to these crystals. So these are differentially removed from the water – which is thus warmed, by their departure (as with boiling, but with the direction of temperature reversed). Again as with boiling: the thermodynamics of freezing involves the specifics and chemistry of the substances in question, and can be quite complicated. For example, water expands on freezing (as one knows from burst pipes!). This – the *anomalous expansion* of water – involves hydrogen bonds between water molecules.

It is easier for water to freeze when there is some existing ice – or even dust – to 'freeze onto', a phenomenon known as nucleation. If rain falls through dust-free air, it may not freeze even though the temperature is well below freezing-point. When this happens, it falls as rain rather than hail or snow – but instantly freezes on impact. The trees then become overloaded with an increasingly thick coating of ice, and fall under the weight – an *ice storm*, common in the USA and Canada.

Q7. *What causes a rainbow?*

C. B. BOYER, *The rainbow: From myth to mathematics*, Princeton UP, 1987 (1st ed. 1959).

The rainbow is formed when the sun is behind the observer, and rain is

falling ahead of the observer. One needs two physical principles:
(a) The law of reflection: when light is reflected at a mirror, the angle of incidence = angle of reflection.
(b) Snel's law of refraction: "mu sin theta = constant", where mu is the refractive index (higher for water than for air, as light travels more slowly in water than in air), and theta is the angle of incidence): the Dutch scientist Willebrord Snel (1581-1625), 1618 and 1621.

The rainbow has a definite 'size'. This is the angle between the line L_1 from the Sun through the observer and any line L_2 from the observer to the arc of the rainbow (any such line gives the same angle, as this arc is circular).

The rainbow is produced when light from the Sun is
(i) *refracted* when it enters falling raindrops in front of the observer (and is bent *towards* the normal);
(ii) *reflected* at the back of the raindrop (which acts as a mirror: angle of incidence = angle of reflection);
(iii) *refracted* again (bent *away* from the normal) when it exits the raindrops. Draw a diagram! See e.g. Boyer, *The Rainbow*, or any decent book on Optics.

A visible effect – the rainbow – is obtained when the angle of deviation has an *extremum* (minimum). For here, many rays will emerge parallel.

Light of different colours (wavelengths) have different refractive indices, so are separated by the two refractions, and one sees the colours of the rainbow. The red end of the visible spectrum subtends an angle of 42.25^o, the violet end an angle of 40.58^o, giving a bow of width c. 1.7^o. This is the *primary rainbow*.

The first qualitatively correct explanation [Week 6, 17th C.] is due to *M. A. de Dominis* (1564-1624) in 1611 (there is also work by Kepler in 1611).

This was taken further by Snel in 1618 and 1621.
René Descartes (1596-1650); *Discours de la Méthode ...*, 1637; (first appendix, *La Géométrie*, cartesian geometry;) second appendix, *La Dioptrique*. This gave a treatment of the rainbow, including an estimate of the angle, 42^o. Descartes did not have calculus, so could not give an analytic solution.
(Sir) Isaac Newton (1642-1727). Newton lectured in Cambridge (1669-71) on the rainbow, and wrote a paper for the Royal Society in 1672 on the composite nature of white light. His *Principia* of 1687 contains all this.

Q8. *What causes a secondary (double) rainbow?*

With a *secondary rainbow*, there are *two* reflections rather than one. The extra reflection both reverses the order of the colours, and makes the bow fainter (light loses intensity when reflected – one can see this on shining a torch beam at a mirror). The secondary bow is larger (adapting the mathematics needed to calculate the angle of the primary bow – see Boyer, etc.

Q9. *What is a sonic boom, and what causes it?*

A sonic boom is a shock wave, caused by a moving body travelling faster than the speed of sound. So the body 'gets ahead of its own sound'. Typical examples are a rifle bullet (hence the characteristic crack/whistle), and a supersonic jet (only military nowadays, since Concorde went out of service).

As the body accelerates through the 'sound barrier', the relevant PDE changes from elliptic (as with Laplace's equation) through parabolic (as with the heat equation) to hyperbolic (as with the wave equation here: d'Alembert, Week 7, 18th C.). The mathematics here became topical again with supersonic flight in the late 20th C. [Week 11].

Q10. *What causes partial reflection of light at a mirror?*

The light ray is a stream of photons – particles, radiation. The mirror is matter. The interaction of matter and radiation is the subject of Quantum electrodynamics (QED). See e.g.

R. P. FEYNMAN, *QED: The strange theory of light and matter*, Princeton UP, 1985.

Newton studied partial reflection of light at a mirror in his *Opticks* of 1704 [Week 6]. We now know it is a *quantum* phenomenon [early 20th C.: Week 10]. Newton had no hope of solving it long before the quantum age began with Planck in 1900. What impressed Feynman is that Newton recognised that there is a deep mystery here. See Week 11 for more on Feynman and QED.

NHB