m3hsoln3.tex

M3H SOLUTIONS 3. 10.2.2017

Q1 *Archimedes' sphere-cylinder theorem.*

Consider the element dA of surface area of the sphere between angles θ and $\theta + d\theta$ (θ is the angle between the line OP from the centre O to the point P on the sphere with the axis of the cylinder). To first order, this is a circular band of radius $r \sin\theta$ and thickness $rd\theta$ [as $r \sin\theta$ is the radius of this circular band]. So

$$dA = 2\pi r^2 \sin\theta d\theta.$$

Integrate over $\theta \in [\alpha, \beta]$:

$$A = 2\pi r^2 \int_\alpha^\beta \sin\theta d\theta = 2\pi r^2 [\cos\alpha - \cos\beta].$$

But this is the area of the part of the cylinder between $\theta = \alpha$ and $\theta = \beta$ [the slice has height $r\cos\beta - r\cos\alpha$].

Q2 *Conics.*

Take the axis L of the cone as the z-axis Oz. Then the radius r of the cone at height z is proportional to z. So the cone has equation of the form

$$x^2 + y^2 = k^2 z^2.$$

The plane Π of section has equation of the form

$$ax + by + cz = d.$$

If Π is horizontal (perpendicular to L), $c = 0$ and the locus of intersection is a circle (whose equation is of the second degree). If not, we can eliminate z between the two equations above. We are left with an equation of the second degree in x and y, as required. [Note that all types of conic can arise in this way, as expected.]

Q3. *Apollonius' 3-line problem.*

If the lines are L_i: $x\cos\alpha_i + y\sin\alpha_i - c_i = 0$, the distance d_i from $P = (x, y)$ to L_i is $PL_i = x\cos\alpha_i + y\sin\alpha_i - c_i$ (to within sign). So the locus of $d_1^2 = cd_2d_3$ is

$$(x\cos\alpha_1 + y\sin\alpha_1 - c_1)^2 = \pm c(x\cos\alpha_2 + y\sin\alpha_2 - c_2)(x\cos\alpha_3 + y\sin\alpha_3 - c_3)$$

(the sign can be determined from one point) – a conic.
(b) *Apollonius' 4-line problem.* Similarly, the locus of $d_1 d_2 = c d_3 d_4$ is

$$(x\cos\alpha_1 + y\sin\alpha_1 - c_1)(x\cos\alpha_2 + y\sin\alpha_2 - c_2) =$$

$$\pm c(x\cos\alpha_3 + y\sin\alpha_3 - c_3)(x\cos\alpha_4 + y\sin\alpha_4 - c_4),$$

again a conic.

Q4 *Focus-directrix property of conics: Pappus.*
Let $F = (x_0, y_0)$ be the focus, $L:\ x\cos\alpha + b\sin\alpha - c = 0$ be the directrix. If $P = (x, y)$, the locus of $PF = e.PL$ is

$$(x - x_0)^2 + (y - y_0)^2 = e^2(x\cos\alpha + y\sin\alpha - c)^2,$$

a conic.

NHB