m3hsoln4.tex

M3H SOLUTIONS 4. 17.2.2017

Q1. (i) Pythagoras' theorem in ΔOMP gives

$$r^2 = u^2 + (\frac{1}{2}s)^2. \qquad (a)$$

Then with $v := r - u$, Pythagoras in ΔMQR gives

$$\begin{aligned} w^2 &:= v^2 + (\frac{1}{2}s)^2 \\ &= (r-u)^2 + (\frac{1}{2}s)^2 = r^2 - 2ru + u^2 + (\frac{1}{2}s)^2 \\ &= 2r^2 - 2ru = 2r(r-u) = 2rv, \end{aligned}$$

by (a).
(ii) The n-gon approximation to the circumference 2π of the unit circle is the perimeter $P_n = ns$; the $2n$-gon approximation is $P_{2n} = 2nw$. Starting with $n = 4$, the unit square with $s = \sqrt{2}$, 6 iterations give

$$P = 6.283153355 \qquad (n = 256).$$

The actual value of 2π correct to 9 places is

$$6.283185308.$$

The approximations increase with n, but one further iteration (on my calculator, a Casio fx-100) gives

$$6.283252652,$$

which is too high, so this is the best we can do by this method.

Q2 *Fibonacci numbers.*

The Fibonacci sequence satisfies the difference equation $u_n - u_{n-1} - u_{n-2} = 0$. The associated *characteristic equation* is $\lambda^2 - \lambda - 1 = 0$, with roots $\frac{1}{2}(1 \pm \sqrt{5})$. So the general solution is $u_n = c_1(\frac{1}{2}(1+\sqrt{5}))^n + c_2(\frac{1}{2}(1-\sqrt{5}))^n$. We can find c_1, c_2 from the initial conditions $u_0 = u_1 = 1$, giving

$$u_n = \frac{1}{2}(1 - 1/\sqrt{5})(\frac{1}{2}(1-\sqrt{5}))^n + \frac{1}{2}(1 + 1/\sqrt{5})(\frac{1}{2}(1+\sqrt{5}))^n.$$

For large n the second term dominates, and the result follows on division.

Q3 *Long division: Fibonacci.*
(i) If x is rational, $x = m/n$ say:
(a) take off its integer part – so reducing to $0 \leq m < n$,
(b) cancel m/n down to its lowest terms.
(ii) Now find the decimal expansion of m/n by the Long Division Algorithm. Let the remainders obtained by $r_1, r_2, \ldots$. The expansion *terminates* if some $r_k = 0$. It *recurs* if some remainder has *already occurred*. As there are only $n - 1$ different possible non-zero remainders, the expansion must terminate (with remainder 0) or recur (with a remainder the first repeat of one of $1, 2, \ldots, n-1$) after at most $n - 1$ places.
(ii) If x is a terminating decimal, x is a rational of the form $n + m/10^k$.
If x is a recurring decimal, say

$$x = n.a_1 \ldots a_k b_1 \ldots b_\ell \ldots b_1 \ldots b_\ell \ldots,$$

x is $n.a_1 \ldots a_k$ (rational, above) $+y$, where writing

$$b := b_1/10 + \ldots b_\ell/10^\ell$$

(rational, above), y is a geometric series with first term $b/10^k$ and common ratio $10^{-\ell}$. So

$$y = b.10^{-k}/(1 - 10^{-\ell}),$$

rational, so x is rational.
Combining with (i): x is rational iff its decimal expansion terminates or recurs.
(iii)
$1/7 = 0.142857$ recurring; $2/7 = 0.285714$ rec.; $3/7 = 0.428571$ rec.;
$4/7 = 0.571428$ rec.; $5/7 = 0.714285$ rec.; $6/7 = 0.857142$ rec.
These are the same six digits in each case, in cyclic order. What we see here is some connection between the denominator 7 and the base 10. For theoretical background here, see e.g. G. H. Hardy & E. M. Wright, *An introduction to the theory of numbers*, 6th ed., OUP, 2008 (1st ed. 1938),
Ch. IX: The representation of numbers by decimals.
But note that from the point of view of Number Theory, the only natural way to expand a real number is as a *continued fraction*: see e.g. Hardy & Wright, Ch. X: Continued fractions.

NHB