

M3H SOLUTIONS 6. 3.3.2017

Q1 Viète's infinite product for π (Francois Viète (1540-1603) in 1593).

(i) The n -gon divides the circle (area π) into n congruent triangles, each of angle $2\pi/n$. Each has area $\frac{1}{2} \sin \pi/n \cos \pi/n = \frac{1}{4} \sin(2\pi/n)$. So

$$A_n = \frac{1}{4}n \cdot \sin(2\pi/n), \quad A_{2n} = \frac{1}{4} \cdot 2n \cdot \sin(\pi/n), \quad A_n/A_{2n} = \cos(\pi/n).$$

Now $A_4 = 2$ (square of side $\sqrt{2}$), and $(\cos \theta = \sqrt{\frac{1}{2}(1 + \cos 2\theta)})$

$$\cos(\pi/4) = 1/\sqrt{2} = \frac{\sqrt{2}}{2}, \quad \cos(\pi/8) = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2 + \sqrt{2}}}{2},$$

$$\cos(\pi/16) = \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2}, \dots$$

As $A_n \uparrow \pi$, Viète's product follows.

Q2 Girard's formula of spherical excess (Albert Girard (1595-1632), *Invention nouvelle en algèbre*, 1629).

On a sphere, a *lune* is the region between two great circles. The ratio of the area of the "A-lune" to that of the sphere is A/π (draw a diagram), and similarly for the B- and C-lunes. If we sum the areas of the three lunes, we cover the area of the sphere, but that of the spherical triangle ΔABC and its antipodal triangle three times (draw a diagram), giving a sum of $S + 4\Delta$ (where S, Δ are the areas of the sphere and triangle). Divide by $S = 4\pi r^2$:

$$\frac{A}{\pi} + \frac{B}{\pi} + \frac{C}{\pi} = 1 + \frac{4\Delta}{4\pi r^2} : \quad \Delta = r^2(A + B + C - \pi).$$

Q3 (Wallis' product for π).

$$\begin{aligned} I_n &= \int \sin^n x \, dx = - \int \sin^{n-1} x \, d \cos x = -\sin^{n-1} x \cos x + \int \cos x \cdot (n-1) \sin^{n-2} x \cos x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \cos^2 x) \, dx : \end{aligned}$$

$$nI_n = -\sin^{n-1}x \cos x + (n-1)I_{n-2}.$$

Passing to the definite integral $J_n := \int_0^{\pi/2} \sin^n x \, dx$ gives

$$J_n = \frac{n-1}{n} \cdot J_{n-2}.$$

So as $\int_0^{\pi/2} dx = \pi/2$, $\int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1$,

$$J_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \quad J_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{2}{3}.$$

Dividing,

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2m}{2m-1} \cdot \frac{2m}{2m+1} \cdot \frac{J_{2m}}{J_{2m+1}}.$$

But as $0 \leq \sin x \leq 1$ in $[0, \pi/2]$, $\sin^{2m+1}x \leq \sin^{2m}x \leq \sin^{2m-1}x$; integrating gives $J_{2m+1} \leq J_{2m} \leq J_{2m-1}$. So

$$1 \leq \frac{J_{2m}}{J_{2m+1}} \leq \frac{J_{2m-1}}{J_{2m+1}} = 1 + \frac{1}{2m} \downarrow 1 : \quad \frac{J_{2m}}{J_{2m+1}} \rightarrow 1 \quad (m \rightarrow \infty).$$

So

$$\frac{\pi}{2} = \lim_{m \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2m}{2m-1} \cdot \frac{2m}{2m+1}.$$

As $2m/(2m+1) \rightarrow 1$, this gives

$$\frac{2^2}{3^2} \cdot \frac{4^2}{5^2} \cdots \frac{(2m-2)^2}{(2m-1)^2} \rightarrow \frac{\pi}{2},$$

which is Wallis' product.

Note.

Take square roots and multiply top and bottom by $2 \cdot 4 \cdots (2m-2) \cdot 2m \cdot \sqrt{2m}$. In the numerator $2^2 \cdot 4^2 \cdots (2m-2)^2 (2m)^2 = 2^{2m} (m!)^2$, giving

$$\frac{2^{2m} (m!)^2}{(2m)! \sqrt{m}} \rightarrow \sqrt{\pi} : \quad \binom{2m}{m} \cdot \frac{1}{2^{2m}} \sim \frac{1}{\sqrt{m\pi}} \quad (m \rightarrow \infty),$$

useful in Probability Theory (Central Limit Theorem for Bernoulli trials: de Moivre-Laplace theorem).

NHB