m3hw8.tex **Week 8. 7 – 10.3.2017**

19th C.

Pierre-Simon de Laplace (1749-1827), Professor at the Ecole Normale and the Ecole Polytéchnique.

Laplace has been called the French Newton. He was a great mathematician, whose name appears in the Laplace transform, Laplace's equation, the Laplacian, etc. He wrote two great books, in quite different areas.
Mécanique Céleste (1799-1825), Vols. I-V (MC)
Laplace's equation (1782/85, 1787/89):

$$\nabla^2 V := \partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial V/\partial z^2 = 0.$$

In electromagnetism (EM theory) and gravitation, the potential V satisfies Laplace's equation if there is no charge density, Poisson's equation (with $4\pi\rho$ on RHS) with charge density ρ. See e.g. NHB, MPC2 Week 10.

Stability of the solar system (1773).

Of MC, Fourier commented "He undertook to compose the Almagest of his age – the *Mécanique Céleste*; and his immortal work carries him as far beyond that of Ptolemy as the analytical science of the moderns surpasses the *Elements* of Euclid."
Volumes I, II (1799): motions of planets; shapes of planets; tides.
Volumes III, IV (1802); Vol. V (1825).
Théorie des attractions des spheroides et de la figure des planètes (1782; included in MC): introduced the idea of a *potential*. For background, see e.g. A. S. RAMSEY, *An introduction to the theory of the Newtonian potential*, CUP, 1940

Laplce's second great book was
Théorie Analytique des Probabilités (1812) (TAP) (2nd ed. 1814, 3rd 1820).

For reasons of space, we will not attempt a full account, but refer to Anders HALD, *A history of mathematical statistics from 1750 to 1930*, Wiley, 1998.
Laplace wrote (Ch. IV of TAP) about the *method of least squares* (due to Legendre and Gauss), achieving the *Gauss-Laplace synthesis*. More technically, he also introduced *Laplace transforms*, and *generating functions* (prototype: $x/(e^x - 1)$ *generates* the Bernoulli numbers B_n as its expansion coefficients).

Carl Friedrich Gauss (1777-1855)

Gauss was arguably the greatest mathematician of all time, certainly (with Archimedes and Newton) one of the greatest three.

Born the son of a stonemason in Braunschweig (Brunswick, Germany), Gauss was a childhood prodigy. At three, he corrected the arithmetic of his father, when reckoning wages as a foreman. At ten, he calculated the sum of the first hundred natural numbers as 5,050 correctly – mentally. His evident genius was recognised,and led to his beng sponsored, from the age of 14 on, by the Duke of Brunswick. Gauss' first mathematical publication, of 1796, concerns the construction of a regular 17-gon, a possibility that had escaped the Greeks. In the tradition of Archimedes, Gauss asked for the regular 17-gon to be carved on his gravestone (whih in fact bears a 17-pointed star, for visual reasons).

Prime Number Theorem (PNT). In 1792, the 15-year old Gauss conjectured PNT (cf. Legendre, Week 7).

Law of Quadratic Reciprocity. Gauss found his 'golden theorem', or 'gem of arithmetic', in 1795, and proved it in 1796 (again, cf. Legendre).

Constructibility of regular n-gons. Gauss showed that a regular n-gon can be constructed by ruler and compass iff $n = 2^k \prod_{i=1}^{j} p_i$, where the p_i are distinct Fermat primes (primes of the form $p = 2^{2^m} + 1$. Recall $p_0 = 3$, $p_1 = 5$, $p_2 = 17$, $p_3 = 257$, $p_4 = 65,537$.

For a nice account of the 17-gon, and its links with Galois theory (below), see e.g. Hardy & Wright, 5.8, or

H. WEYL, *Symmetry.*

The Journals

From 1796 to 1814, Gauss kept a private journal, in which he listed 146 of his discoveries. The journal was published only in 1898; meanwhile Gauss' failure to publish (reminiscent of Newton, whom he admired above all others) led him into priority disputes (e.g. with Legendre over the Method of Least Squares, below). It is hard for a modern observer to understand this almost pathological reticence. One aspect is that Gauss admired the rigour of the Greeks so much that he wished to publish only perfect proofs: *Pauca, sed matura* (few, but ripe) was his motto. Another possibility was his dislike (shared with Newton) of controversy: his work on non-Euclidean geometry, for instance, and perhaps on complex numbers, was far enough ahead of its time to have run the risk of controversy on publication.

Fundamental Theorem of Algebra (1799)

Gauss' doctoral thesis (in Latin: 'A new proof that every polynomial of one variable can be factored into real factors of the first or second degree') was published in 1799.

Despite its name, this result is a theorem of *analysis*, not of *algebra*. Its proof was less rigorous than Gauss' usual standard: he assumed properties of continuous functions later proved by Bolzano.

Disquisitiones arithmeticae (1801) (Researches in number theory), Gauss' master work on number theory.

I. Congruences in general.

II. Congruences of the first degree. Fundamental Theorem of Arithmetic.

III. Residues of powers. Fermat's theorem.

IV. Congruences of the second degree. Law of Quadratic Reciprocity.

V. Quadratic forms. Diophantine problems: $ax^2 + 2bxy + cy^2 = d$.

VI. Applications, VII: Division of the circle: $x^n = 1 \pmod p$; construction of the regular n-gon.

The *Disquisitiones* also contains the determinant (by that name, and long before matrices!).

The Orbit of Ceres

On 1 January 1801, Piazzi observed, in Palermo, a new planetoid, which he kept under observation for 41 days (when it disappeared behind the Sun). The most famous astronomical problem of the day was to predict its reappearance by calculating its orbit from the data. Gauss re-discovered Legendre's method of least squares for this purpose, correctly predicting Ceres' reappearance in December 1801. He published his work, on Ceres, Pallas (another planetoid) and least squares in

Theoria motus corporum coelestium in sectionibus conicus solem ambientem (1809) (TM; Eng. tr. *Theory of the motion of heavenly bodies ...*, 1857).

Gauss was now famous, and was appointed Director of the new Observatory in Göttingen in 1804, and Professor of Astronomy there in 1807. He remained in Göttingen for the rest of his life.

Gauss' patron, the Duke of Brunswick, was mortally wounded at the Battle of Auerstadt in 1805. Gauss was much affected by his death, that of his first wife, and the financial burden of the enforced transfer of Göttingen to Westphalia.

Least Squares. The priority dispute with Legendre.

Recall Legendre's published account of 1805, and Gauss' TM of 1809.

Legendre's treatment presented and named the method, but did not link it to Probability Theory. Gauss, however, *linked least squares with the normal law.* He said in TM 'On the other hand, our principle (principium nostrum), which we have made use of since the year 1795, has lately been published by Legendre ...'. No doubt Gauss was telling the literal truth. Understandably, however, Legendre felt slighted that Gauss should thus seek to claim priority for something first published by Legendre, and protested, first in private correspondence, and in 1820 publicly in a Second Supplement to his 1805 book.

Much more important than this dispute over priority, however, is the impact of Gauss' work on Statistics, which is profound. It is the basis of the *linear model* of Statistics; see e.g. Bingham & Fry. In modern terminology, the likelihood is

$$L = (2\pi)^{-\frac{1}{2}n} \sigma^{-n} \exp\{-\frac{1}{2} \sum_i (y_i - \sum_j x_{ij}\theta_j)^2 / \sigma^n\}.$$

Thus the Method of Least Squares (minimise the sum) becomes here the Method of Maximum Likelihood (Fisher, 20th C.; cf. Daniel Bernoulli).

The Gauss-Laplace Synthesis

Following Gauss' TM of 1809, there were now two ways in which the normal law entered the theory:
(i) via CLT (the de Moivre-Laplace limit theorem);
(ii) through the link with Least Squares.
Laplace recognised the importance of (i) and (ii) together in a Supplement added to a paper of his in 1810, and incorporated this *Gauss-Laplace synthesis* into his TAP of 1812. See e.g.
S. M. STIGLER, *The history of statstics: The measurement of uncertainty before 1900*, Harvard UP, 1986, Ch. 4.

Geodesy

The shape of the Earth was still an open problem at that time, and Gauss became involved in the triangulation of Hannover (Göttingen was then in Hannover, ruled by King George III), with a view to finding the best approximating ellipsoid to the Earth's shape, in 1820. In 18213 Gauss completed the field-work, developing the heliotrope (an instrument involving reflecting sunlight by mirrors)for this purpose. [The pre-euro 10 DM note shows Gauss and the normal curve on the face and the heliotrope and triangulation on the reverse.]

Differential Geometry

Disquisitiones generales circa superficies curvas (1827) (General investigations on curved surfaces).

Differential geometry is vital for e.g. relativity (20th C.).

The book covers parametric representation of surfaces; curvilinear coordinates; conformal property; Gaussian curvature K; the Gauss-Bonnet theorem. For geodesic triangles with angles A, B, C, $A+B+C-\pi = \int KdS$, with dS the 'element of surface area'. For a sphere (constant positive curvature), one recovers Girard's formula of spherical excess. The plane has zero curvature, and $A+B+C=\pi$. The case $K<0$ involves *non-Euclidean geometry*, a possibility that Gauss hinted at in a letter of 1799 to his friend Wolfgang Bolyai, father of Johan (Janos) Bolyai (co-discoverer of non-Euclidean geometry, with Lobachevski).

Magnetism

In collaboration with Wilhelm Weber (1804-1891), who became Professor of Physics at Göttingen in 1831, Gauss constructed the first electromagnetic telegraph in 1833. Gauss' work in magnetism is commemorated in the name of the unit of magnetism (1881). See Whittaker, I Ch. VIII. [1]

Hypergeometric series

$$F(a,b;c;z) := \sum_{n=0}^{\infty} \frac{a(a+1)\dots(a+n-1).b(b+1)\dots(b+n-1)}{n!c(c+1)\dots(c+n-1)} z^n;$$

for background, see e.g. Whittaker & Watson, Ch. XIV.

AM-GM and Elliptic Functions.

Recall the 'AM-GM inequality' of elementary Analysis: arithmetic mean $\geq$ geometric mean. If $a > b > 0$, $a_0 := a$, $b_0 := b$; $a_1 := \frac{1}{2}(a+b)$, $b_1 := \sqrt{ab}$,

$$a_n := \frac{1}{2}(a_{n-1} + b_{n-1}), \qquad b_n := \sqrt{a_{n-1}b_{n-1}},$$

Gauss showed (c. 1800; found in his posthumous papers) that both sequences converge (quite rapidly): $a_n \to M$, $b_n \to M$, where $M = M(a,b)$ is the 'arithmetico-geometric average' of a, b. This involves an elliptic integral:

$$\frac{1}{M(1-x,1+x)} = \int_0^{\pi} \frac{d\theta}{\sqrt{1-x^2\cos^2\theta}}.$$

Remarkably, this has led, in the computer age, to algorithms for calculating π to millions of decimal places. For background, see

[1] In WWII, magnetic mines were dropped by parachute to block the sea lanes into British ports. Damage to surviving ships showed pressure damage, suggested that the mine did not explode in contact. This suggested that the mines were triggered magnetically, confirmed when a bomb disposal expert dismantled one and found a dial calibrated in Gauss. Anti-magnetic counter-measures were promptly and successfully taken.

J. M. BORWEIN & P. B. BORWEIN, *Pi and the AGM*, Wiley, 1987.
Aside: Calculating pi: π is now known to some ten trillion digits (W).
The complex plane $\mathbb{C}$.

We take so much for granted the modern view of the complex plane $\mathbb{C}$ ($z = (x, y) \leftrightarrow x + iy \leftrightarrow re^{i\theta}$) that it as well to realise how recent this is. We owe this *Argand diagram* to Wessel, Argand and Gauss: Caspar Wessel (1745-1818) in 1799 (Danish; tr. 1895), Gauss in 1831, and
Jean-Robert Argand (1768-1822): *Essai sur une manière de répresenter les quantités imaginaires dans les constructions géométriques*, 1806: geometric representation; name 'complex number'.
Numerical Analysis. Gauss did a great deal of calculation, and was interested in numerics. His name is remembered in *Gaussian elimination* (Numerical Linear Algebra – the systematic way to solve simultaneous linear equations), and *Gaussian quadrature* (the nth approximation to $\int f(x)w(x)dx$ by a sum $\sum f(x_i)w(x_i)$ over the zeros x_i of the nth orthogonal polynomial w.r.t. the weight w: Gauss, *Werke*, Vol. 3).
Vector calculus: "grad, div, curl/Stokes, Gauss, Green": see e.g. NHB, MPC2 Dram. Pers., and C. F. Gauss (1813): Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata, *Commentationes societatis regiae scientiarium Gottingensis recentiores*, 2: 355-378; Gauss considered a special case of the theorem; see the 4th, 5th, and 6th pages of his article.
Biographies. See e.g.
Tord HALL, *Carl Friedrich Gauss, A biography*, MIT Press, 1970 (Swedish, 1965);
W. K. BÜHLER, *Gauss, a biographical study*, Springer, 1981.

Augustin-Louis Cauchy (1789-1857), Professor at the Ecole Polytéchnique and later the Sorbonne.

Cauchy was the second most prolific mathematician of all time (after Euler). He was the creator of Complex Analysis. He was also one of the main creators of modern rigorous Analysis.
Cours d'Analyse de l'Ecole Polytéchnique (1821)
Résumé des lecons sur le calcul infiniésimal (1823)
Lecons sur le calcul différentiel (1829).
Cauchy General Principle of Convergence

This result – that a sequence is convergent iff it satisfies the Cauchy condition – is in his *Cours d'Analyse* of 1821, p. 125. This involves the

completeness property of $\mathbb{R}$ and $\mathbb{C}$, to which we return later in connection with rigorous construction of the real number system in 1872, and in the 20th C. with topology.
Cauchy's Theorem

$$\int_\gamma f(z)dz = 0$$

for an analytic function f and a contour Γ, if f has no singularities inside Γ. This is in a MS of Cauchy of 1825, published 1874.

From NHB, M2P3, Dramatis Personae (which contains a detailed history of Complex Analysis to M2P3 level):
Augustin Louis Cauchy (1789-1857): introduces Complex Analysis (1825-1829) [I.0]; Cauchy's General Principle of Convergence [I.2.8]; Root Test [I.2.11]; Cauchy-Riemann equations [II.2]; Cauchy's theorem [II.5]; Cauchy's integral formulae [II.6]; Cauchy's inequalities [II.6]; Cauchy-Taylor theorem [II.7]; Cauchy's Residue Theorem [II.11]; Cauchy density [III.4].

Joseph Fourier (1768-1830)
Théorie Analytique de la Chaleur (1822) (*The analytical theory of heat*, Dover, 1955); *Mémoire sur la propagation de la chaleur*, read (Inst. Fr.) 1807.

The propagation of heat in a medium is governed by the *heat equation*: with k the thermal diffusivity, the temperature u satisfies

$$\nabla^2 V := \partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial V/\partial z^2 = k^{-1}\partial u/\partial t.$$

The theory of *Fourier series* – representing a function $f(\theta)$ by a trigonometric series $\sum(a_n \cos n\theta + b_n \sin n\theta)$ – is closely connected with the study of heat propagation, and both were developed by Fourier, in his pioneering book (1822) and earlier (1807 etc.). Similarly for Fourier integrals.
As mentioned earlier, Fourier became a Prefect under Napoleon. See e.g.
John HERIVEL, *Joseph Fourier: The man and the physicist*, OUP, 1975;
Ivor GRATTAN-GUINNESS, *Joseph Fourier, 1768-1830*, MIT Press, 1972.

Simeon-Denis Poisson (1781-1840)
Traité de mécanique, Vol. I (1811), Vol. II (1833); *Poisson brackets* (see e.g. Goldstein, *Classical mechanics*, 8.5).
Poisson's equation (1812): $\nabla^2 V = -4\pi\rho$.
Recherches sur la probabilité des jugements en matière criminelle et en

matière civile (1837): *Poisson distribution* $P(\lambda)$, giving mass $e^{-\lambda}\lambda^k/k!$ for $k = 0, 1, 2, \ldots$. This dominates the probability and statistics of *counts*, e.g. of accidents, deaths etc. in the insurance/actuarial professions.

Carl Gustav Jacobi (1804-1851)

Jacobi was a superb mathematician, who died tragically young. We shall say little about his work, as much of it was so advanced it is still more postgraduate material than undergraduate. We meet his name first in the *Jacobian*, or functional determinant $|\partial u_i/\partial x_j|$ in the 'chain rule' for calculus of several variables (1829, 1841).

Theta functions: Whittaker & Watson, Ch. XXI. One of the shortest proofs of the *functional equation* for the Riemann zeta function uses Jacobi's identity for the theta function (NHB, M3P16, III.8 L25).

Partitions (Number Theory): Hardy & Wright, Ch. XIX.

Elliptic functions: Whittaker & Watson, Ch. XX, XXII (cf. Abel, below).

Dynamics: *Vorlesungen über Dynamik* (1866, posth.). Hamilton-Jacobi theory, a bridge from classical to quantum mechanics. Jacobi's identity for Poisson brackets: $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$.

Niels Henrik Abel (1802-1829)

The Norwegian mathematician Abel had a tragically short life, dogged by poverty, ill health and misfortune. His best work was unappreciated by Gauss; a MS of his was lost by Cauchy; his work on elliptic functions was almost simultaneous with that of Jacobi, etc.

On the algebraic resolution of equations (1824). Proof of the insolubility of the quintic (completing work of 1799 by Paolo Ruffini (1765-1822). Abel's work on the quintic is closely linked to with Galois theory (below), whence the term *abelian* for a commutative group.

Elliptic functions: double periodicity, 1827-8. See Jacobi (above), and W& Ch. XX, XXII.

Abel's test for convergence: partial summation. See e.g. NHB, M3P16, I.3 L2-3 and Handout).

Abel's continuity theorem for power series, *J. Math.* 1 (1826), 311-339: if $f(x) = \sum_0^\infty a_n x^n$ converges at x_0,$f(x) \to f(x_0)$ as $x \uparrow x_0$. There is no converse, but there are many *partial converses*, called *Tauberian theorems*. See e.g. NHB, M3P16, Handout, Tauberian theorems.

For Abel's life and times, see e.g.

Arild STUBHAUG, *Niels Henrik Abel and his times: Called too soon by flames afar*, Springer, 2000 (Norwegian, 1996).

Evariste Galois (1811-1832)

Galois' short life, and tragic death in a duel aged 21, are described in e.g. E. T. BELL, *Men of Mathematics*, Ch. 20.

Galois was a key figure in the history of Algebra. He introduced the term *group* in 1830, and considered *fields* obtained by adjoining roots of polynomial equations. This generalises the introduction of the complex field (or plane) $\mathbb{C}$ by formally adjoining to the real field (or line) $\mathbb{R}$ $i := \sqrt{-1}$, a root of the (previously rootless) polynomial $x^2 + 1$. There are whole books (and courses) on Galois Theory; for a textbook account, see e.g.
P. M. COHN, *Algebra*, Vol. 2 (2nd ed.), Wiley, 1989, Ch. 3.

J. P. G. Lejeune Dirichlet (1805-1859): Professor of Mathematics at Berlin (1845-55); Director of the Observatory at Göttingen, in succession to Gauss.

Dirichlet became famous for his proof (1837) that there are infinitely many primes in any arithmetic progression (on the integers, with first term and interval coprime). With this, and his introduction of Dirichlet series (1838, 1839), Analytic Number Theory (ANT) begins.
Dirichlets test (for convergence of series, a complement to Abel's), 1828.
Riemann, a pupil of Dirichlet, who worked on convergence of Fourier series, considered Dirichlet the founder of this field.
NHB, M3P16, Dram. Pers.: I.3; Dirichlet convolution, II.3; Dirichlet eta function (alternating zeta function), II.1; Dirichlet Hyperbola Identity, II.9; Dirichlet Pigeonhole Principle, 1837, III.10.5.
Vorlesungen über Zahlentheorie (1863, posth.): ideals in algebraic number theory.

G. F. Bernhard Riemann (1826-66), Professor of Mathematics at Göttingen (1857); Director of the Observatory at Göttingen (1859), in succession to Gauss and Dirichlet.
Über die Hypothesen, welche der Geometrie zu Grunde liegen (On the hypotheses which lie at the foundations of geometry), Habilitationsschrift, Göttingen, 1854 (publ. 1867. posth.).

Here Riemann introduces (in modern terminology) *Riemannian manifolds*. Loosely, a *manifold* is a space that is locally Euclidean but not (necessarily) globally so (e.g., a sphere is locally flat but globally round). Thus in Euclidean 3-space, the distance formula (Pythagoras' theorem) can be expressed in local form as $ds^2 = dx^2 + dy^2 + dz^2$. In an n-dimensional Riemannian manifold, this is replaced by $ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$, with the quadratic form $g = g_{ij}$ positive definite. With the idea of distance comes that of shortest path, or *geodesic* (e.g., arcs of great circles on the sphere, whence the name). Riemannian manifolds give the mathematical framework needed for Einstein's General Theory of Relativity (20th C.).

The Laplacian generalises from Euclidean space to the *Laplace-Beltrami operator* for manifolds (see below for Beltrami).

Riemann surfaces

Riemann also introduced *Riemann surfaces*, important in complex analysis, geometry and topology. Prototype: the complex logarithm $z \mapsto \log z$ is not a function (is many-valued), but becomes a function if its range is changed to a suitable Riemann surface R (an 'infinite stack' of complex planes $\mathbb{C}$, spliced along the positive half-line, so that as one performs a revolution in the plane $\mathbb{C}$ one 'goes up one' in the stack, as in a spiral staircase. For the extensive background, see e.g.

S. K. DONALDSON, *Riemann surfaces*, OUP, 2011;

A. F. BEARDON, *A primer of Riemann surfaces*, LMS LNS 78, CUP, 1984.

Riemann Mapping Theorem

The *Riemann Mapping Theorem* (Dirichlet's Göttingen thesis of 1851, but with a defective proof) states that if U is a non-empty, proper, simply connected open subset of $\mathbb{C}$, there exists a biholomorphic (bijective and holomorphic) mapping $f : U \mapsto \mathbb{D}$, the open unit disk. This extends to Riemann surfaces. Recall the *Riemann sphere* $\mathbb{C}^*$ (or extended complex plane under stereographic projection). A simply connected open subset of a Riemann surface R is biholomorphic to one of $\mathbb{C}^*$, $\mathbb{C}$ and $\mathbb{D}$ (Uniformisation Th.).

Integration; trigonometric series

The Riemann integral (a formalisation of the 'Sixth Form integral' with epsilons) appears in Riemann's paper of 1854 on trigonometric (Fourier) series. See

A. ZYGMUND, *Trigonometric series*, CUP, 1959, Ch. IX.

Riemann zeta function; Riemann Hypothesis (RH)

Über die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatshefte Berliner Akad.* (1858/60), 671-680.

This astonishing paper, only 10 pages long and written as an inaugural lecture at Göttingen in 1859, is all that Riemann ever wrote on Analytic Number Theory, yet it changed the subject forever. It does not prove the PNT, but shows the deep links between the distribution of primes and the complex zeros of the Riemann zeta function $\zeta(s)$ in the *critical strip* $0 < Re\ s < 1$. RH, still open, says that they all lie on the *critical line* $Re\ s = \frac{1}{2}$. If true, this would greatly increase our knowledge of the distribution of primes. For background, see e.g. NHB, M3P16.

NON-EUCLIDEAN GEOMETRY

Nikolai Ivanovich Lobachevski (1793-1856), Professor of Mathematics at Kazan.

Lobachevski had as a colleague J. M. Bartels (1769-1836), a former teacher of Gauss. Whether through knowledge by Bartels of Gauss' interest (c. 1799, above), or independently, Lobachevski worked in the 1820s on the Parallel Postulate, and became convinced – rightly – that it could not be deduced from the other Euclidean postulates. He then took the logical but epoch-making step of substituting for it an alternative postulate (through any point C not on line AB can be drawn *more than one* line in the plane ABC not meeting AB. He did this in two Russian papers (of 1829 and 1835/37, both now lost), and: *J. für Math.* **17** (1937), 295-320;
Geometrische Untersuchungen zur Theorie der Parallellinien (1840);
Pangéométrie (1855).

Janos Bolyai (1802-1860), son of Farkas (Wolfgang) Bolyai (1775-1856).

Gauss wrote to Farkas Bolyai on 17.12.1799, explaining his thinking on the Parallel Postulate. He worked on non-Euclidean geometry (he introduced the term) in the first two decades of the 19th C. In connection with his work on triangulation, he even performed an experiment, measuring the angles of a triangle formed by three mountain peaks. But, as usual, he published nothing on the subject.

The science of absolute space (1832). Janos Bolyai published a 26-page *Appendix: Scientiam absolute veram exhibens* to his father's book *Tentamen juventutem studiosam in elementa matheseos* [Essay on the elements of mathematics for studious youths]. His ideas are also mentioned in his father's book *Kurzer Grundriss eines Versuchs* (1851).

Farkas had despaired of proving the parallel postulate, and warned his son against taking up the subject. But Janos announced his successful construction of (a) non-Euclidean geometry in letter to his father of 3.11.1823. His father then advised him to publish it, in case others did so. The credit for non-Euclidean geometry is thus shared between Lobachevski and Bolyai, who had the courage of their convictions and published their work (independently, and roughly simultaneously).

Eugenio Beltrami (1835-1899)

Saggio di interpretazione della geometria non-euclidea. *Gior. di Mat.* **6** (1968), 284-312.

Beltrami constructed a non-Euclidean geometry on a *pseudosphere*, a space of constant negative curvature, the surface of revolution of a tractrix (the locus of a mass dragged by a string whose end moves along a line).

The more usual model for non-Euclidean geometry nowadays is Poincaré's (later), on a half-plane or disc.

PROJECTIVE GEOMETRY

Desargues' projective geometry of the Brouillon projet (1639) was brought to modern form in the 19th C., largely by the French school:
Gaspard Monge (1746-1818), Professor at the Polytéchnique;
Charles Jules Brianchon (1785-1864); *Victor Poncelet* (1788-1867).

Poncelet (MS c. 1812, publ. 1862-4, Works I, II) emphasised *duality* in Projective Geometry: in two dimensions, one may interchange the words 'point' and 'line'; in three dimensions one may interchange 'point' and 'plane', leaving 'line' the same. Poncelet wrote his projective geometry in two columns, the right obtained from the left by duality. A prime example of duality is *Pascal's theorem* on hexagons inscribed to conics, and *Brianchon's theorem* on hexagons circumscribed about conics – *discovered* by duality.
The Platonic solids. The cube and octahedron are dual; the dodecahedron and icosahedron are dual; the tetrahedron is self-dual.

GROUPS AND GEOMETRY

Felix Klein (1849-1925), Professor of Mathematics at Erlangen (1872).

Klein's inaugural lecture at Erlangen set out his *Erlanger Programm.* This was a scheme to study geometry in terms of invariance under groups of transformations. Thus to Klein: Euclidean geometry is the study of properties invariant under the action of the Euclidean geroup (of rigid motions, or change of coordinate system – rotations and translations). Similarly with 'Euclidean' replaced by 'projective', and (an intermediate case) by 'affine'. [The term affine, meaning 'alike', expresses the sense in which, say, two ellipses are like each other, but unlike a hyperbola.]

Klein also studied non-Euclidean geometry. He introduced the terms *elliptic geometry* (positive curvature – extending spherical geometry) and *hyperbolic geometry* (negative curvature: an alternative to Beltrami's representation on the pseudosphere). Klein's model of hyperbolic geometry was later joined by that of Poincaré (below). See e.g.
A. F. BEARDON, *The geometry of discrete groups*, GTM 91, Springer, 1983.
Sophus Lie (1842-1889), a Norwegian mathematician, worked on continuous transformation groups (initially with Klein, though they quarreled later); *Theorie der Transformationsgruppen*, Vol 1-3 (1888-93, with F. Engel).