M3H/M4H/M5H HISTORY OF MATHEMATICS: EXAMINATION SOLUTIONS, 2017

The solutions given below are intended to be indicative rather than prescriptive. Most of what follows is taken from the teaching material, with full detail (dates etc.) included. This is for completeness and for information; students are not expected to memorise dates accurately (except for the *Principia* in 1687 – "*1066 and all that*".

Section A: answer 5 questions out of 10; 10 marks each.

Q1. *The pentagram.*

The *star pentagram* (below) was known to the Pythagoreans, who flourished in the 6th C. BC in Croton (now S. Italy). Linked with this is the *golden section* ('the section' in antiquity; the term golden section is due to Kepler, 17th C.: the ratio of the sides of a rectangle such that if a square on the smaller side is removed, the remaining rectangle is similar to the original one. This ratio is held to be visually pleasing in art and architecture. **[2]**

Star pentagram and golden section (Euclid Book 6, Prop. 30).

In a regular pentagon $ABCDE$ of side a, join up each vertex to its two opposite vertices. The resulting figure is the *star pentagram*, and contains an inner pentagon $A'B'C'D'E'$ say (with A' the vertex opposite A, etc.), of side $a-b$ say (so $b = AD' = AC'$, etc.).

$\Delta AD'B$ is isosceles ($AD' = BD'$ by symmetry), so $\angle D'AB = \angle D'BA, = \theta$ say. Write $\theta' := \angle EBD$. Then $2\theta + \theta' = 3\pi/5$ (at B, the interior angle is $3\pi/5$, as the complementary exterior angle is $2\pi/5$). Angle $\angle AD'B = \pi - 2\theta$ (angle-sum in $\Delta AD'B$). So the complementary angle $\angle AD'C' = 2\theta$. Triangle $\Delta AD'C'$ is isosceles, by symmetry; its angle-sum gives $4\theta + \theta' = \pi$. Eliminating θ', $\pi - 4\theta = 3\pi/5 - 2\theta$: $\theta = \pi/5$, and then $\theta' = \pi/5$. So the interior angles are *trisected*. **[2]**

(i) Triangles ΔEAB and $\Delta EC'A$ are similar (both isosceles, with angles $\pi/5, \pi/5, 3\pi/5$), with sides $a, a, a+b$ and b, b, a. So

$$\phi := \frac{a}{b} = \frac{a+b}{a} = 1 + \frac{1}{\phi} : \quad \phi^2 - \phi - 1 = 0 : \quad \phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$$

(we take the $+$ sign in $\pm$ since $\phi > 0$). **[2]**

(ii) The outer and inner pentagons have sides a, $a-b$, whose ratio is

$$(a-b)/a = 1 - 1/\phi = 2 - \phi = \frac{1}{2}(3 - \sqrt{5}). \qquad \textbf{[2]}$$

(iii) Dropping the perpendicular $C'C''$ from C' to AE, $\cos(\pi/5) = \frac{1}{2}a/b = \frac{1}{2}\phi$:

$$\phi = 2\cos(\pi/5).$$

Dropping the perpendicular AA'' from A to $C'D'$, we get a right-angled triangle with angle $\pi/10$ at A, hypotenuse b and opposite side $\frac{1}{2}(a-b)$. So

$$\sin(\pi/10) = \frac{\frac{1}{2}(a-b)}{b} = \frac{1}{2}(\phi - 1) : \qquad \phi = 1 + 2\sin(\pi/10). \qquad \textbf{[2]}$$

[Seen – lectures and problems]

Q2. *The heliocentric theory.*

Aristarchus (of Samos, fl. c. 280 BC)

T. L. HEATH: *Aristarchus of Samos: The ancient Copernicus* (1913; 1981).

By observations made of eclipses, etc., Aristarchus was able to estimate the relative sizes of the earth, sun and moon. His estimates, though highly inaccurate by modern standards, were a good deal better than previous ones. He published his results in a book, 'On the dimensions and distances of the sun and moon'.

Both Archimedes (Dreyer, 138-8) and Plutarch (Dreyer, 138-140) gave detailed accounts of Aristarchus' views on the universe (the modern phrase 'solar system' is perhaps too coloured by hindsight here), in which they assert that Aristarchus pictured the earth as rotating about the sun – the *heliocentric* system of today. As for his book, Dreyer (p. 136) says flatly 'This treatise does not contain the slightest allusion to any hypothesis on the planetary system ...'; Boyer asserts (p. 180) that the book takes a geocentric view. But Heath, in his book on Aristarchus (p. iv) states that:

'... there is still no reason to doubt the unanimous verdict of antiquity that Aristarchus was the real originator of the Copernican hypothesis'.

Thus Aristarchus is a figure of tremendous importance; he has claims to be regarded as 'the ancient Copernicus'. **[3]**

Nicholas Copernicus (1473-1543) of Thorn (Niklas Koppernigk of Torun);

De revolutionibus orbium coelestium, 1543.

This work revolutionised astronomy by expounding the *heliocentric theory* – that the earth and other planets revolve around the sun. With Copernicus, the modern period of astronomy begins. See Dreyer Ch. XIII for a detailed account of this epoch-making achievement (and Dreyer Ch. VI and Week 2 for a discussion of Aristarchus and the heliocentric theory). **[4]**

Galileo Galilei (1564-1642); *Astronomy.*

Galileo invented (as well as an air thermometer) a telescope. With this, he began observations in 1609, observing

(i) the Mountains of the Moon; (ii) the four Moons of Jupiter;

(iii) the phases of Venus (incompatible with the geocentric system, since this shows that Venus orbits round the Sun).

This provided crucial observational support for the Copernican theory.

The Two Chief Systems (1632). Written in the form of a dialogue between three characters, this book supported the Copernican heliocentric theory, and brought Galileo into conflict with the Inquisition. **[3]** [Seen – lectures]

Q3. *Alexandria.*
Euclid (of Alexandria; fl. c. 300 BC). **[2]**

Ptolemy I succeeded Alexander the Great in Egypt, and consolidated his rule by 306 BC. He established the Museum at Alexandria, the leading academy (in effect, university) of its day. Here Euclid taught, and wrote, around 300 BC, the most successful mathematics textbook ever written, the *Elements*. This is divided into thirteen books: Books I-VI on elementary plane geometry, VII-IX on the theory of numbers, X on incommensurables, XI-XIII on solid geometry. He also wrote the *Optics*, the *Phaenomea* (on spherical geometry), *Division of Figures* and other works, now lost.

Euclid (we speak of the author and his Elements interchangeably from now on) expounds mathematics – particularly geometry – in what is essentially the modern style – *axiomatically.* One begins with a list of *axioms* (as few as possible, and as 'obvious', or 'innocuous', as possible). From these, one deduces, by rigorous proof, *theorems* (as many as possible, as 'non-obvious' as possible).

Following the conquest of Greece by Rome in 146 BC, Greek mathematics went into a decline, followed by a partial recovery. Its principal exponents came much later, and were mainly based in Alexandria, e.g.:
Menelaus, c. 100 AD, **[1]**
Ptolemy, fl. c. 127-150 AD, **[1]**
Heron, 3rd C. AD, **[1]**
Diophantus, fl. c. 250 AD, **[1]**
Pappus (of Alexandria, fl. c. 290 AD) (B 11.7-10, Heath II, Ch. XIX).

So far as his original mathematics is concerned, Pappus is remembered for:
Focus-directrix(-eccentricity) theorem for conics (Coolidge, *Conics and quadrics*, 8-13): If F (focus) is a point, L (directrix) is a line, e (eccentricity) is a positive constant, the locus of a point P such that its perpendicular distance PL to the line L satisfies $PF = e.PL$ is a conic section.

Pappus was the last of the major Greek geometers, ending a tradition spanning some eight centuries. He was also the last great Alexandrian mathematician (though we should also remember the martyrdom of *Hypatia*, 415 AD). No other centre of learning has ever dominated mathematics for as long (the six centuries from Euclid to Pappus) as did Alexandria. **[2]**

Our knowledge of the history of Greek (and Alexandrian) mathematics is largely due to *Proclus* (410-485), and his *Eudemian Summary*, of the account of *Eudemus* (of Rhodes, fl. c. 320 BC, a student of Aristotle). **[2]**
[Seen – Lectures]

Q4 *Galileo* (1564-1642)

'Galileo was the first truly modern scientist – the first whose outlook and methods would not be out of place even today. He made important discoveries in astronomy and mechanics, but his greatest achievement was the creation of the experimental-mathematical method that has lain at the basis of all progress in physical science since his time'.

Professor of Mathematics at Pisa, 1589. He demonstrated that heavy and light weights fall at the same speed by dropping them from the Leaning Tower of Pisa. His experimental approach to science brought him into conflict with the Church (below). He was driven out of Pisa in 1592, going to Padua as Lecturer in Mathematics. He eventually returned to Pisa, becoming Philosopher and Mathematician to the Grand Duke of Florence. **[2]**

Astronomy.

Galileo invented a telescope, and began observations in 1609, observing (i) the Mountains of the Moon; (ii) the four Moons of Jupiter; (iii) the phases of Venus (incompatible with the geocentric system, since this shows that Venus orbits round the Sun). These discoveries gave the Copernican theory crucial observational support. **[2]**

The Two Chief Systems (1632). Written (like Plato's dialogues) in the form of a dialogue between three characters, this book supported the Copernican heliocentric theory, and brought Galileo into conflict with the Inquisition. Under threat of torture, in 1633 he was forced to disavow it. The (Roman Catholic) Church admitted it was wrong to condemn Galileo in 1992. **[2]**

The Two New Sciences (1638): on mechanics. He showed that

(i) a body falling under gravity does so with constant acceleration;

(ii) the trajectory of a projectile is a parabola.

Although the ancient Greeks had an excellent knowledge of conics, they were capable of asserting that projectiles describe arcs of circles! This illustrates both Greek limitations and the power of Galileo' new scientific method. **[2]**

Infinite sets. Galileo noticed the characteristic property of an infinite set: it can be put into one-one correspondence (bijection) with a proper subset of itself (e.g., $\mathbb{Z} \leftrightarrow 2\mathbb{Z}$ under $n \leftrightarrow 2n$). This was taken up in the 19th C. by Dedekind. **[1]**

Epitaph. Galileo is reported to have withdrawn his forced recantation on his deathbed with the words 'E pur si muove' ('Eppur si muove' – 'And still it moves'), which, apocryphal or not, may serve as an epitaph to one of the giants of science. **[1]**

[Seen – lectures]

Q5. *Huygens.*

Christiaan Huygens (1629-1695) was Dutch, a pupil of Frans van Schooten (1615-1660), Professor of Mathematics at Leiden. Huygens moved from the Netherlands to Paris in 1668, when the Académie des Sciences was founded. [**1**]

Traité de la lumière (publ. 1690; read to the French Academy, 1678).

Huygens advocated the *wave theory of light*; the work also covers reflection and refraction, and polarisation. [**1**]

Pendulum clock, 1656 (Galileo is said to have proposed this in 1641).

Horologium oscillatorium (1673): [**1**]

Centripetal force for circular motion. Principle of Conservation of Energy. This book was an important precursor of Newton's *Principia*.

Telescopes: Observation of the rings of Saturn. [**1**]

Thermometers: suggested 0 and 100 for freezing and boiling points – the *Centigrade* scale (long before Anders Celsius (1707-1744) in 1742). [**1**]

Huygens' Principle: [**1**]

Light travels along paths of shortest time. Kline (579-582) discusses the history of this idea, from the Greeks (Heron) through Fermat and Huygens to Euler.

The cycloid. He knew from his work on the pendulum clock that the period of a pendulum is only approximately constant, but depends on the displacement. He sought a curve such that a particle sliding on it has *exactly* the same period regardless of displacement, and showed this was a *cycloid.* [**1**]

Envelopes and wavefronts. His work on the wave theory of light led Huygens to regard propagation of light as a continuous process of 'ripple formation': subsidiary wavelets released at time t from, through their envelope, the wavefront at time $t + dt$. For background, see e.g.

J. L. SYNGE, *Geometrical optics: An introduction to Hamilton's method.* Cambridge Tracts in Math. 37, CUP, 1937. [**1**]

De ratiociniis in ludo aleae (1657) (On reasoning in dice games). This was the first important book on Probability Theory (recall Cardano's *De ludo aleae* was written c. 1526 but only published in 1663). [**1**]

Gravitation. Huygens opposed Newton's Law of Gravity, which he regarded as unphysical ('action at a distance'). For background, see e.g.

Sir Edmund WHITTAKER, *A history of the theories of the aether and electricity.* Dover, 1989 (Nelson, 1910/1951), Volume I, Ch. 1: The theory of the aether to the death of Newton. [**1**]

[Seen – lectures]

Q6. *The electromagnetic theory of light.*
James Clerk Maxwell (1831-1879) [Whittaker, Ch. VIII: Maxwell].
A treatise on electricity and magnetism, Vol. 1, 2, OUP, 1891/1998.
Maxwell's Equations. If E, H are the electric intensity and the magnetic field in ES (electrostatic) units, cE in EM (electromagnetic) units, Maxwell's equations (in a vacuum) are

$$\text{div } E = 0, \quad \text{curl } E = -c^{-1}\partial H/\partial t; \quad \text{div } H = 0, \quad \text{curl } H = c^{-1}\partial E/\partial t. \quad [\mathbf{2}]$$

Whitaker (p. 245) writes: 'In this memoir (of 1865) the physical importance of the operators curl and div first became evident. These operators had, however, occurred frequently in the writings of Stokes ...'. Applying curl (recall curl curl = grad div - ∇^2, $= -\nabla^2$ here):

$$-\nabla^2 E = \text{curl curl}E = -\frac{1}{c}\frac{\partial}{\partial t}(\text{curl}H) = -\frac{1}{c2}\frac{\partial^2 E}{\partial t^2} : \qquad \nabla^2 E = c^{-2}\partial^2 E/\partial t^2,$$

and similarly

$$\nabla^2 H = c^{-2}\partial^2 H/\partial t^2. \qquad [\mathbf{2}]$$

This is the *wave equation*, for propagation of E, H with velocity c, the ratio of EM to ES units. This was known experimentally (c. 3×10^{10} cm/sec., c. 186,000 miles/sec.) to be (approx.) the *speed of light*. Thus, *electromagnetic forces are propagated with the speed of light.* This suggested (correctly) to Maxwell that *light waves are electromagnetic.* Recall the modern *electromagnetic spectrum*: in increasing order of wavelength, ..., x-rays, ultra-violet, visible spectrum, infra-red, radio waves, ... **[2]**

To do the mathematics above, one needs:
(i) *The wave equation*, the prototypical hyperbolic (linear, 2nd order) PDE: J. d'Alembert (1717-1783) in 1746.
(ii) *Vector calculus: grad, div and curl*: the divergence theorem, or Gauss' theorem (1813); Green's theorem (George Green (1793-1841; *Essay on magenetism and electricity*, 1828); Stokes' theorem (the curl theorem), 1854. **[2]**

Maxwell's EM theory of light builds on the work of Faraday on electromagnetic induction; together these constitute the two greatest advances of 19th C. Physics. From them flow the essentials of our modern life: electric power; radio, television, etc. **[2]**
[Seen – lectures]

Q7. *The calculus of variations.*
Christiaan Huygens (1629-1695). **[2]**
Huygens' Principle: Light travels along paths of shortest time. This idea can be traced from the Greeks (Heron – as paths of shortest distance) through Fermat and Huygens to Euler.
Leonhard Euler (1707-1783); born in Basel, a pupil of Jean Bernoulli and colleague of Daniel Bernoulli. **[3]**
Calculus of Variations (CoV). To maximise (or minimise) an integral

$$I := \int_a^b F(x, y, y')dx \qquad (y' = dy/dx)$$

with respect to variation in the function $y = y(x)$: Euler showed in 1744 that the solution satisfies the '*Euler-Lagrange equation*'

$$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0. \tag{EL}$$

Example: the Brachistrochrone. One formulation is: a particle slides under gravity along a smooth curve from A to B. Find the curve for which the time taken is a minimum (brachos = short, brachistos = shortest + chronos = time, Greek). The problem (in one form or another) was solved by Bernoulli (Jean and/or Jacques; also by Newton and Leibniz, *Acta Eruditorum*) in 1696: the solution is a cycloid (the locus of a point on a circle rolling without slipping on a plane – also often used for the arches of bridges). This problem led to the development of the CoV. It may easily be solved by (EL).
Joseph-Louis Lagrange (1736-1813) (Turin; Berlin Academy;
French Academy). **[3]**
Calculus of Variations (CoV). This was Lagrange's earliest (and possibly best) work. In 1755 he wrote to Euler about his work on CoV. Euler generously held up publication of his own work, so that Lagrange's work – which Euler thought superior – should get full credit, and advised Frederick to bring Lagrange to Berlin. The 'Euler-Lagrange equations' date from this time. The name *Calculus of Variations* is due to Euler in 1766
Sir William Rowan Hamilton (1805-1865), Professor of Astronomy, Trinity College Dublin, 1827-65. **[2]**
Hamilton's Principle (1834): With L the Lagrangian, $\int Ldt$ is an extremum. This includes the *Principle of Least Action*, which can be traced back to Maupertuis (1744).
[Seen – lectures and problems]

Q8. *Projective geometry.*
Girard Desargues (1591-1661) and *Projective Geometry.* **[4]**

Desargues' first important book was *La Perspective* (1636). This led him on to his introduction of projective geometry in his *Brouillon projet (d'une atteinte aux evenements des rencontres d'une cone avec un plan)* (1639) *Rough draft (of an attempt to deal with the outcome of a meeting of a cone with a plane)*. As background, recall:

(i) Many results in geometry concern only *incidence properties* (whether lines meet, point lie on a line, etc.); these are preserved under projection.

(ii) Often one has to qualify statements because of exceptional cases involving 'infinity' – e.g., two lines in a plane meet in a point (unless parallel). Recall *perspective* (vanishing point = 'point at infinity').

All this led to Projective Geometry, in which it is incidence properties rather than metrical ones that count. Here one works *projectively*, using *homogeneous coordinates* (in which point in a plane has three coordinates rather than two, determined up to a constant multiple). This powerful tool allows much simplification, but involves a thorough-going change of viewpoint.

The *Brouillon Projet*, though pioneering, had little impact at the time: too far ahead of its time; too badly written; too few copies.

Projective methods in geometry, together with analytic (= coordinate) and synthetic (= classical) methods, complete the main tools needed to treat the geometric problems studied up to that time.

Conics. Projective methods allow a simple interpretation of conics as sections of circular cones by planes: the conic are the projections of circles. **[1]**

Projective geometry is of great practical importance: it is the basis of computer graphics, hence of virtual reality etc.

Blaise Pascal (1623-1662).

Essay pour les coniques (1640) (one page!) Pascal's theorem (on hexagons inscribed in a conic), inspired by Desargues' work. **[1]**

Charles Jules Brianchon (1785-1864); *Victor Poncelet* (1788-1867).

Poncelet (MS c. 1812, publ. 1862-4, Works I, II) emphasised *duality* in Projective Geometry: in two dimensions, one may interchange the words 'point' and 'line'; in three dimensions one may interchange 'point' and 'plane', leaving 'line' the same. A prime example of duality is *Pascal's theorem* on hexagons inscribed to conics, and *Brianchon's theorem* on hexagons circumscribed about conics – *discovered* by duality. **[3]**

The Platonic solids. The cube and octahedron are dual; the dodecahedron and icosahedron are dual; the tetrahedron is self-dual. **[1]** [Seen – lectures]

Q9. *The Fundamental Theorem of Algebra.*

Carl Friedrich Gauss (1777-1855)

Gauss' doctoral thesis (in Latin: 'A new proof that every polynomial of one variable can be factored into real factors of the first or second degree') was published in 1799.

Despite its name, this result is a theorem of *analysis*, not of *algebra*. Its proof was less rigorous than Gauss' usual standard: he assumed properties of continuous functions later proved by Bolzano. **[3]**

Augustin-Louis Cauchy (1789-1857), Professor at the Ecole Polytéchnique and later the Sorbonne.

Cauchy's *Cours d'analyse* (Ecole Polytéchnique, 1821) contains a proof of the Fundamental Theorem of Algebra: every complex polynomial of degree n has n complex roots (counted according to multiplicity). **[3]**

The modern proof uses *Liouville's theorem* (Joseph Liouville (1809-1882), lectures in 1847 – actually published by Cauchy in 1844): an entire (i.e. holomorphic throughout the complex plane $\mathbb{C}$) bounded function is constant. Thus it took over twenty years before the full power of Cauchy's new subject of Complex Analysis was properly brought to bear on the Fundamental Theorem of Algebra – incidentally, revealing in so doing that the result, being a theorem in Analysis, is a misnomer. **[2]**

N. H. Abel (1802-29): Insolubility of the quintic (1829).

Although the quintic has five roots, by the Fundamental Theorem of Algebra above, Abel showed that – in contrast to polynomials of degree up to four, which can be solved, as Cardano showed – quintics are *not soluble by radicals*: there can be *no* formula/algorithm/method for expressing the roots in terms of the coefficients. Similarly for polynomials of higher degree. So there is a fundamental split: polynomial equations are soluble by radicals for degree up to 4, but not for degree 5 or higher. **[2]**

[Seen – lectures]

Q10. *Mathematics and Biology: Darwin, Mendel and Fisher.*
Charles DARWIN (1809-1882): *On the Origin of Species by means of Natural Selection*, 1859 – *The Origin of Species*). **[2]**

Darwin based his theory on decades of thought and observation. He delayed publishing his work for years, for fear of upsetting his wife, who was deeply religious. Of course, Darwinian evolution contradicted the account of the Creation in the Bible (Genesis). His theory was accordingly attacked, e.g. in the famous debate between Bishop Wilberforce and T. H. Huxley ('Darwin's bulldog'), after whom the building containing the Imperial College Mathematics Department is named.

Darwinian evolution by natural selection is widely regarded as the greatest single idea in the history of human thought.

Gregor MENDEL (1822-1884): **[2]**
Experiments on plant hybridization, 1866).

Mendel was a Bohemian monk. His most famous experiment was on the numbers of smooth and wrinkled peas produced when different strains of pea are crossed: these occur in simple arithmetic proportions. His work was published in an obscure journal, where it went unnoticed and was largely forgotten, until it was rediscovered in 1900. It laid the foundations for the modern science of *genetics*.

R. A. (Sir Ronald) FISHER (1890-1962). **[3]**

Fisher was the greatest statistician ever, and one of the greatest geneticists. He did his greatest work at the Rothamsted Experimental Station in Harpenden, Herts in the 1920s (studying crop yields, for which he developed much of modern Statistics). He was Galton Professor of Eugenics at UCL (1933-43) and Arthur Balfour Professor of Genetics at Cambridge (1943-59).

The two foundations of modern biology are the Darwinian theory of natural selection and Mendelian genetics. It was thought at first that Mendelian genetics and Darwinian natural selection were incompatible, but this is not so; the two were synthesized by three people: the American Sewall Wright (1889-1988) and the Englishmen Fisher (above) and J. B. S. Haldane (1892-1964). The resulting Darwin-Mendel synthesis is the basis of modern biology.

The relevant mathematics used *Markov chains* (*A. A. Markov* (1856-1922) in 1907). These give models for stochastic processes (random phenomena unfolding with time) in which there is no memory. The *Fisher-Wright model* of math. genetics is one of the most important classical Markov chains. **[3]**
[Seen – lectures]

Section B. Answer two questions; 25 marks each

Q1. *The Renaissance and its mathematics*

Background **[3]**

The Renaissance, or re-birth, marks the end of the Dark Ages and the re-emergence of European culture dormant since the classical period. The history of mathematics, and science, will be primarily concerned with Europe (and later, its offshoots in America) from now on.

The Renaissance (the term dates back only to 1855, in Michelet's *Histoire de France*) is a broad term covering the 14th-17th centuries, but is regarded as having begun in Florence in the 14th C. One factor here was the role of the Medici family, who were originally bankers before going into politics. Banking and finance (of which more below) flourished on contact between civilisations, which tended to have a cross-fertilising effect; also, the Medici's money enabled them to become great patrons of the arts. Later, the influx of Greek scholars after the Fall of Constantinople in 1453, bringing with them many texts and much learning, was also an important factor.

Perspective **[5]**

As we have seen, perspective was known (at least in part) in the ancient world, but was then lost.

Filippo Brunelleschi (1377-1446) discovered the main principle of perspective – the use of vanishing points – and convinced his fellow-artists of this in a famous experiament of 1420 involving the chapel outside Florence Cathedral.

Leon Battista Alberti (1404-72), *Della pictura* (1435, printed 1511) gave the first written account of perspective.

Piero della Francesca (1410-92), *De prospectivo pingendi* (c. 1478). In his book, and in his painting, Piero della Francesca did much to popularise perspective, which spread throughout the Western art world.

Leonardo da Vinci (1452-1519); *Trattato della pittura*. Leonardo is usually regarded as the personification of Renaissance genius. He was a prolific inventor, an artist who wrote on perspective, and a mathematician.

Albrecht Dürer (1471-1528) of Nuremburg; *Investigations of the measurement with circles and straight lines of plane and solid figures* (1525-1538, German and Latin). Like Leonardo, Dürer was both a mathematician and an artist. He adopted perspective after visiting Italy.

Printing and books **[2]**

An important turning-point was the invention of the printing-press. Movable type was introduced by Johannes Gutenberg (1395-1468) of Mainz, in his

Bible of 1455. The first printed book in English was produced by William Caxton in Bruges in 1476 (he then moved to London and established the first printing press in Britain). Printing and books enormously increased the scope for the rapid dissemination of knowledge.
Universities **[3]**

Learning was now dominated by the universities, focussing on mediaeval Latin for theology etc. and Latin translations from the Arabic in science. Printing and books led to more emphasis on Greek culture, both in literature and in science. The Greek classics in both could now be read in the original, translated directly (from the Arabic), printed, and distributed widely.
Regiomontanus (1436-76) **[2]**

Born Johann Müller of Königsberg (now Kaliningrad in Russia), he was known as Regiomontanus (king's mountain, Latin, Königsberg, German). He completed a new Latin translation of Ptolemy's *Almagest* (begun by Peuerbach), which was mathematically superior to previous versions.
De triangulis omnimodis (1464). Probably influenced by the work of the Arab mathematician Nasir Eddin, this was the first major European work on trigonometry, and helped as a subject, independent of astronomy.
Nicholas Chuquet (fl. c. 1500) **[2]**
Triparty en la science des nombres, 1484: the most important European mathematical text since the *Liber Abaci*.
Luca Pacioli (1445-1514) **[2]**
Summa de arithmetica, geometrica, proportioni et proportionalita, 1494 (B 15.7). This was an elementary text (more influential than the *Triparty*) on arithmetic, algebra and geometry. It is notable for *double-entry book-keeping*, and use of the *decimal point*.
Geronimo Cardano (1501-76); *Ars Magna*, 1545. **[3]**

Cardano's *solution of the cubic* (published here in 1545) marks 'the beginning of the modern period in mathematics' (del Ferro and Tartaglia also solved the cubic). Cardano introduced *complex numbers* in the *Ars Magna*.
Nicholas Copernicus (1473-1543) of Thorn (Niklas Koppernigk of Torun, Poland); *De revolutionibus orbium coelestium*, 1543. **[3]**

This work revolutionised astronomy by expounding the *heliocentric theory*. With Copernicus, the modern period of astronomy begins.

The Renaissance evolved gradually into the early modern period; we regard complex numbers and the heliocentric theory as good markers for the transition. The term Renaissance is surprisingly late (19th C.).
[Seen – lectures]

Q2 *Differential equations*

Early history **[3]**

Study of differential equations (DEs) goes back to the discoverers of the differential and integral calculus, Isaac Newton (1642-1727) and Gottfrfied Leibniz (1646-1716). Newton solved a DE by infinite series in 1676, only 11 years after his discovery of the fluxional form of differential calculus in 1665. This was published in 1693, the year Leibniz first published a DE (his differential calculus was published in 1684).

Jean Bernoulli (1667-1748) introduced separation of variables in 1694-97.

Jakob Bernoulli (1654-1705) introduced and solved Bernoulli's equation.

Leonhard Euler (1707-1783). **[5]**

Institutiones calculi differentialis (1755);

Institutiones calculi integralis (1768-70, Vol I-III)

Euler began the systematic study of ordinary DEs (ODEs – one independent variable). He introduced the idea of an *integrating factor*, now standard. He studied *linear DEs with constant coefficients*, reducing them to the familiar characteristic polynomial. For linear DEs, he studied the homogeneous and non-homogeneous cases, showing in particular that the arbitrary constants belong with the *general solution (GS)* of the homogeneous equation. Thus one obtains the familiar

$$\text{GS (non-homogeneous)} = \text{GS (homogeneous)} + \text{particular integral (PI)}.$$

Euler also introduced and solved the *Euler DE*

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = 0.$$

The first partial differential equation (PDE) to be studied was thkat of the vibrating string. This second-order PDE, the *wave equation* in one dimension,

$$\partial^2 u/\partial t^2 = c^2 \partial^2 u/\partial x^2,$$

GS $u(x,t) = f(x+ct) + g(x-ct)$ (representing waves propagating forwards and backwards) was solved by Euler and d'Alembert (1717-83) in 1747.

In higher dimensions, the wave equation becomes

$$\nabla^2 V = c^{-2} \partial^2 V/\partial t^2$$

($\nabla^2 := \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, the Laplacian).

Euler method for numerical solution of DEs, now used also for stochastic differential equations (SDEs).

P. S. de Laplace (1749-1827). **[3]**
Laplace's equation: $\nabla^2 V = 0$ (1789: memoir on Saturn's rings).
S.-D. Poisson (1781-1840). **[2]**
Poisson's equation: $\nabla^2 V = -4\pi\rho$ (1812).
This generalises Laplace's equation for the (gravitational) potential V from empty space to the presence of matter.
J. Fourier (1768-1830). **[3]**
Théorie analytique de la chaleur (1822).
Heat equation: $\nabla^2 V = k^{-1}\partial V/\partial t$ (k: thermal diffusivity).
This describes the flow of heat in a conducting body.

Linear second-order PDEs. **[5]**
Poisson's/Laplace's equation: $\nabla^2 V = -4\pi\rho$ [$= 0$]: *elliptic*, relevant to *potential theory* (gravitational or electromagnetic).
Heat equation, $\nabla^2 V = k^{-1}\partial V/\partial t$: *parabolic.*
Wave equation, $\nabla^2 V = c^{-2}\partial^2 V/\partial t^2$: *hyperbolic.*

Much of 19th C. *mathematical physics* is concerned with properties of these PDEs or their relatives – e.g.
Equation of telegraphy: $\partial^2 V/\partial x^2 = CL\partial^2 V/\partial t^2 + CR\partial V/\partial t$
(Sir William Thomson, Lord Kelvin (1824-1907); translatlantic telephone cable, 1866).

Much of the 19th C. work on *special functions* (*Bessel functions*, *Legendre polynomials*, etc.) grew from the study of ODEs derived from such PDEs, by techniques such as separation of variables. *Eigenvalue problems* were also discussed at length (Sturm-Liouville theory, etc.).
20th C.
Aeronautics. Subsonic flight is described by an elliptic PDE. Supersonic flight is described by a hyperbolic PDE: *shock wave.* **[2]**
Navier-Stokes equation. **[2]**

The Navier-Stokes equation for viscous fluids, ubiquitous today, is due to *Claude-Louis Navier* (1785-1863) in 1822 and *Sir George Stokes* (1819-1963) in 1845. It is a semi-linear PDE, of the form

$$\partial u/\partial t + (u.\nabla)u - \nu\nabla^2 u = -\nabla h.$$

[Seen – lectures]

Q3 *Solution of polynomial equations.*
Egypt: Linear equations are found in the Rhind (or Ahmes) Papyrus, c. 1650 BC. **[1]**
Mesopotamia: Some quadratic equations were studied c. 2000 BC (also some cubics). **[1]**
Diophantus of Alexandria, c. 250 AD: For positive rational solutions, Diophantus dealt with quadratic equations completely, and some cubics. **[1]**
India: Brahnagupta (c. 628 AD): General solution of quadratics. **[1]**
al-Khwarizmi (d. 850), *Al-Jabr wa al-Muqabala*. **[1]**

This book synthesised the Mesopotamian, Greek and Hindu algebra traditions, and was the first to do so. **[1]**
Omar Khayyam (c. 1100): Algebra – quadratics and cubics. **[1]**

Thus pre-modern work on the solution of polynomial equations was restricted to real (and sometimes, to positive) roots. Within this framework, it is impossible to account for the fact that a polynomial of degree n (recall that $n \leq 3$ here) may not have n real (or positive) roots).

Turning to the modern European period:
Gerolamo (or Geronimo) Cardano (1501-1570); *Ars Magna*, 1545.

Cardano's *solution of the cubic* (published here in 1545) marks 'the beginning of the modern period in mathematics'. Scipione del Ferro (c. 1465-1526), Professor of Mathematics at Bologna, solved the cubic but did not publish his results. Niccolo Tartaglia (c. 1500-1557) (b. Niccolo Fontana; Tartaglia = stammerer), knowing of del Ferro's solution, found one himself, by 1541. Predictably, this led to a priority dispute with Cardano.
Complex numbers.

Cardano, in Ch. 37 of *Ars Magna*, solves

$$x(10 - x) = 40,$$

obtaining roots $5 \pm \sqrt{-15}$, and notes that

$$(5 + \sqrt{-15})(5 - \sqrt{-15}) = 25 - (-15) = 40.$$

Thus 'Without having fully overcome their difficulties with irrational and negative numbers, the Europeans added to their problems by blundering into what we now call complex numbers' (Kline). Though complex numbers were not properly assimilated into mathematics till much later, they enter the stage with *Ars Magna*.

Also: 'whenever the three roots of a cubic are real and non-zero, the

Cardan-Tartaglia formula leads inevitably to square roots of negative numbers'. **[3]**

Carl Friedrich Gauss (1777-1855)

Fundamental Theorem of Algebra (1799)

Gauss' doctoral thesis (in Latin: 'A new proof that every polynomial of one variable can be factored into real factors of the first or second degree') was published in 1799.

Despite its name, this result is a theorem of *analysis*, not of *algebra*. Its proof was less rigorous than Gauss' usual standard: he assumed properties of continuous functions later proved by Bolzano. **[3]**

Augustin-Louis Cauchy (1789-1857), Professor at the Ecole Polytéchnique and later the Sorbonne.

Cauchy's *Cours d'analyse* (Ecole Polytéchnique, 1821) contains a proof of the Fundamental Theorem of Algebra: every complex polynomial of degree n has n complex roots (counted according to multiplicity). **[3]**

The modern proof uses *Liouville's theorem* (Joseph Liouville (1809-1882), lectures in 1847 – actually published by Cauchy in 1844): an entire (i.e. holomorphic throughout the complex plane $\mathbb{C}$) bounded function is constant. Thus it took over twenty years before the full power of Cauchy's new subject of Complex Analysis was properly brought to bear on the Fundamental Theorem of Algebra – incidentally, revealing in so doing that the result, being a theorem in Analysis, is a misnomer. **[3]**

N. H. Abel (1802-29): Insolubility of the quintic (1829).

Although the quintic has five roots, by the Fundamental Theorem of Algebra above, Abel showed that – in contrast to polynomials of degree up to four, which can be solved, as Cardano showed – quintics are *not soluble by radicals*: there can be *no* formula/algorithm/method for expressing the roots in terms of the coefficients. Similarly for polynomials of higher degree. So there is a fundamental split: polynomial equations are soluble by radicals for degree up to 4, but not for degree 5 or higher. **[3]**

Evariste Galois (1811-1832).

Galois was the first to study *field extensions* systematically. This is the key to the *algebraic closure* of the complex plane $\mathbb{C}$, but not of the real line $\mathbb{R}$: a polynomial of degree n with coefficients in $\mathbb{C}$ has n roots in $\mathbb{C}$, but one with coefficients in $\mathbb{R}$ need not have n roots in R. **[3]**

[Seen – lectures]

Q4 *Number Theory*

Euclid of Alexandria; Elements, c. 300 BC. **[3]**

Book VII: Elementary number theory.

Prop. 2: h.c.f., by the Euclidean algorithm.

Prop. 24: a, b coprime to n implies ab coprime to n.

Book IX: Primes.

Prop. 20. There are infinitely many primes.

Proof (Modern version). Let $p_1, p_2, \cdots, p_n$ be the first n (the n smallest) smallest in the list of primes. Write $N := 1 + p_1 p_2 \cdots p_n$. Then no p_i divides N. So by the Fundamental Theorem of Arithmetic, there is another prime p_{n+1} which does divide N. So the list can always be extended.

Diophantus of Alexandria; Arithmetica, c. 350 AD, Books I-XIII (VII on lost). **[2]**

Book VI contains a treatment of *indeterminate* equations (with more unknowns than equations, so with infinitely many solutions), in which one seeks *integer* solutions. This has developed into the subject of *Diopohantine equations.*

Pierre de Fermat (1601-1665). **[3]**

Fermat was the founder of modern number theory, inspired by Diophantus's Arithmetica (translated 1621).

(i) 'Method of infinite descent' for proving irrationality.

(ii) 'Fermat's theorem': if p is prime, $a^p \equiv a \bmod p$.

(iii) Fermat numbers, $F_n := 2^{2^n} + 1$ (Fermat conjectured these were all prime; F_1 to F_4 are, but Euler showed that F_5 is composite).

(iv) Fermat's conjecture: if $n > 2$, $x^n + y^n = z^n$ has no solutions (x, y, z all non-zero integers) ($n = 2$: Pythagorean triples – there are infinitely many of these!). The problem was the most famous unsolved problem in mathematics, until the conjecture was proved by (Sir) Andrew Wiles (1963-) in 1995.

Leonhard Euler (1707-1783). **[3]**

Euler products: $\zeta(s) := \sum_1^\infty 1/n^s = \prod_p 1/(1 - p^{-s})$ (p prime).

$\zeta(2) = \pi^2/6$ (Basel problem).

Euler's totient function: $\phi(n) :=$ number of integers k, $1 \leq k \leq n$, coprime to n. If $n = \prod p_i^{n_i}$ (FTA), $\phi(n) = n \prod (1 - 1/p_i)$.

Euler's theorem: If y is coprime to n, $y^{\phi(n)} \equiv 1 \bmod n$.

A.-M. Legendre (1752-1833). **[2]**
Essai sur la théorie des nombres (1797-98), Vols I, II – the first book(s) devoted entirely to number theory. Here he conjectured:
The Law of Quadratic Reciprocity, later proved by Gauss (below);
The Prime Number Theorem (PNT): If $\pi(x)$ is the number of primes $p \le x$,

$$\pi(x) \sim x/\log x \quad (x \to \infty); \quad \text{equivalently,} \quad p_n \sim n \log n \quad (n \to \infty).$$

C. F. Gauss (1777-1855). **[2]**
Gaus conjectured PNT in 1792 (aged 15!); he proved the Law of Quadratic Reciprocity in 1795 (publ. 1796).
Disquisitiones Arithmeticae (1801): Gauss's master work on number theory.

P. G. L. Dirichlet (1805 - 1854). **[3]**
Dirichlet's theorem (1837): There are infinitely many primes in every arithmetic progression in $\mathbb{N}$. From NHB, M3P16, Dramatis Personae:
Dirichlet series, 1838, 1839; Dirichlet's test; Dirichlet convolution; Dirichlet's Hyperbola Identity.

G. F. Riemann (1826-66). **[3]**
Über die Anzahl der Primzahlen unter einer gegeben Grössen (1858).

The series $\zeta(s) := \sum_1^\infty 1/n^s$ goes back to Euler, but is called the *Riemann zeta function* in honour of Riemann. He showed the importance of taking s *complex*, $s = \sigma + it$ (of course, Riemann had Cauchy's creation of Complex Analysis in the late 1820s to help him!). He introduced the famous *Riemann Hypothesis (RH)*, still open: $\zeta(s) = 0$, $s = \sigma + it$, $t \neq 0$ implies $\sigma = \frac{1}{2}$, and showed its relevance to the distribution of primes.

The zeta function, and Complex Analysis, played key roles in the proof of PNT in 1896 by (independently!) *C. J. de la Vallée Poussin* (1866-1962) and *J. Hadamard* (1865-1963). **[2]**

In the 20th C., Analytic Number Theory was turned into a systematic subject by *Edmund Landau* (*Handbuch der Lehre von der Primzahlen*, 1909). The PNT was proved by elementary means (no Complex Analysis) in 1937 by Atle Selberg (1917-2007) and Paul Erdős (1913-1996). **[2]**
(There is no intention to neglect *Algebraic Number Theory* here, but we did not cover it.)
[Seen – lectures] N. H. Bingham