HISTORY OF MATHEMATICS: MOCK EXAMINATION SOLUTIONS, 2013.

The solutions given below are intended to be indicative rather than prescriptive. Most of what follows is taken from the teaching material, with full detail (dates etc.) included. This is for completeness and for information; students are not expected to memorise dates accurately (except for the *Principia* in 1687 – "*1066 and all that*".

Section A: answer 5 questions out of 10; 10 marks each.

Q1. *The formula of spherical excess.*
Albert Girard (1590-1633) of Flanders (B 16.4)
Invention nouvelle en l'algèbre (1629): Negative and imaginary roots of polynomials; irreducible case of the cubic; sums of roots, of squares of roots, etc. Girard also conjectured the Fundamental Theorem of Algebra.

The book also contains *Girard's formula*, or the *formula of spherical excess.* For a spherical triangle with angles A, B, C, $A + B + C > \pi$, and $A+B+C-\pi$ is called the *spherical excess.* If the sphere has radius r, then the area Δ of the triangle is given by

$$\Delta = r^2(A + B + C - \pi).$$

Note: if the area is small for fixed r, then so is the excess. So for small spherical triangles on the earth's surface, the sum of the angles is approximately π. This just says that we may neglect the Earth's curvature for triangles small in relation to the Earth, and this is of course used in map making (recall the trig points on OS maps!). **[4]**

On a sphere, a *lune* is the region between two great circles. The ratio of the area of the "A-lune" to that of the sphere is A/π (draw a diagram), and similarly for the B- and C-lunes. If we sum the areas of the three lunes, we cover the area of the sphere, but that of the spherical triangle ΔABC and its antipodal triangle three times (draw a diagram), giving a sum of $S + 4\Delta$ (where S, Δ are the areas of the sphere and triangle). Divide by $S = 4\pi r^2$:

$$\frac{A}{\pi} + \frac{B}{\pi} + \frac{C}{\pi} = 1 + \frac{4\Delta}{4\pi r^2} : \qquad \Delta = r^2(A + B + C - \pi). \qquad \mathbf{[3]}$$

C. F. Gauss (1777-1855) and *Differential Geometry.*
Disquisitiones generales circa superficies curvas (1827).

This covers parametric representation of surfaces; curvilinear coordinates; conformal property; Gaussian curvature K; the Gauss-Bonnet theorem (extended by Pierre Bonnet (1819-1892)in 1848). For geodesic triangles with angles A, B, C, $A + B + C - \pi = \int K dS$, with dS the 'element of surface area'. For a sphere (constant positive curvature), one recovers Girard's formula of spherical excess. The plane has zero curvature, and $A + B + C = \pi$ (Euclid, Book I Prop. 32). The case $K < 0$ involves *non-Euclidean geometry*, a possibility that Gauss hinted at in a letter of 1799 to Wolfgang Bolyai, father of Johan (Janos) Bolyai (co-discoverer of non-Euclidean geometry, with Lobachevski). **[3]**
[Seen – lectures]

Q2. *The mathematics of the rainbow.*

The rainbow is formed when the sun is behind the observer, and rain is falling ahead of the observer. One needs two physical principles:
(a) The law of reflection: when light is reflected at a mirror, the angle of incidence = angle of reflection.
(b) Snel's law of refraction: "mu sin theta = constant", where mu is the refractive index (higher for water than for air, as light travels more slowly in water than in air), and theta is the angle of incidence): the Dutch scientist Willebrord Snel (1581-1625), 1618 and 1621. **[1]**

The rainbow has a definite 'size'. This is the angle between the line L_1 from the Sun through the observer and any line L_2 from the observer to the arc of the rainbow (any such line gives the same angle, as this arc is circular).

The rainbow is produced when light from the Sun is
(i) *refracted* when it enters falling raindrops in front of the observer (and is bent *towards* the normal);
(ii) *reflected* at the back of the raindrop (which acts as a mirror);
(iii) *refracted* again when it exits the raindrops.

A visible effect – the rainbow – is obtained when the angle of deviation has an *extremum* (minimum). For here, many rays will emerge parallel. **[1]**

Light of different colours (wavelengths) have different refractive indices, so are separated by the two refractions, and one sees the colours of the rainbow. The red end of the visible spectrum subtends an angle of 42.1^o, the violet end an angle of 42.1^o. This is the *primary rainbow.* **[1]**

Sometimes one can see a *secondary rainbow*, with *two* reflections rather than one: larger, fainter, and with the order of the colours *reversed.* **[1]**

The first qualitatively correct explanation is due to *M. A. de Dominis* (1564-1624) in 1611 (there is also work by Kepler in 1611). **[1]**

This was taken further by Snel in 1618 and 1621. **[1]**

René Descartes (1596-1650); *Discours de la Méthode ...*, 1637; (first appendix, *La Géométrie*, cartesian geometry;) second appendix, *La Dioptrique.* This gave a treatment of the rainbow, including an estimate of the angle, 42^o. Descartes did not have calculus, so could not give an analytic solution. **[1]**
(Sir) Isaac Newton (1642-1727). Newton lectured in Cambridge (1669-71) on the rainbow, and wrote a paper for the Royal Society in 1672 on the composite nature of white light. His *Principia* of 1687 contains all this. **[1]**

Source: C. B. BOYER, *The rainbow: From myth to mathematics*, Princeton UP, 1987 (1st ed. 1959).
[Mentioned in lectures, though less detail was given there.]

Q3. *The pentagram.*

The *star pentagram* (below) was known to the Pythagoreans, who flourished in the 6th C. BC in Croton (now S. Italy). Linked with this is the *golden section* ('the section' in antiquity; the term golden section is due to Kepler, 17th C.: the ratio of the sides of a rectangle such that if a square on the smaller side is removed, the remaining rectangle is similar to the original one. This ratio is held to be visually pleasing in art and architecture. **[2]**

Star pentagram and golden section (Euclid Book 6, Prop. 30).

In a regular pentagon $ABCDE$ of side a, join up each vertex to its two opposite vertices. The resulting figure is the *star pentagram*, and contains an inner pentagon $A'B'C'D'E'$ say (with A' the vertex opposite A, etc.), of side $a-b$ say (so $b = AB' = AE'$, etc.).

$\Delta AD'B$ is isosceles ($AD' = BD'$ by symmetry), so $\angle D'AB = \angle D'BA, = \theta$ say. Write $\theta' := \angle EBD$. Then $2\theta + \theta' = 3\pi/5$ (at B, the interior angle is $3\pi/5$, as the complementary exterior angle is $2\pi/5$). Angle $\angle AD'B = \pi - 2\theta$ (angle-sum in $\Delta AD'B$). So the complementary angle $\angle AD'C' = 2\theta$. Triangle $\Delta AD'C'$ is isosceles, by symmetry; its angle-sum gives $4\theta + \theta' = \pi$. Eliminating θ', $\pi - 4\theta = 3\pi/5 - 2\theta$: $\theta = \pi/5$, and then $\theta' = \pi/5$. So the interior angles are *trisected*. **[2]**

(i) Triangles ΔEAB and $\Delta EC'A$ are similar (both isosceles, with angles $\pi/5, \pi/5, 3\pi/5$), with sides $a, a, a+b$ and b, b, a. So

$$\phi := \frac{a}{b} = \frac{a+b}{a} = 1 + \frac{1}{\phi}: \quad \phi^2 - \phi - 1 = 0: \quad \phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$$

(we take the + sign in $\pm$ since $\phi > 0$). **[2]**

(ii) The outer and inner pentagons have sides a, $a-b$, whose ratio is

$$(a-b)/a = 1 - 1/\phi = 2 - \phi = \frac{1}{2}(3 - \sqrt{5}). \qquad \textbf{[2]}$$

(iii) Dropping the perpendicular $C'C''$ from C' to AE, $\cos(\pi/5) = \frac{1}{2}a/b = \frac{1}{2}\phi$:

$$\phi = 2\cos(\pi/5).$$

Dropping the perpendicular AA'' from A to $C'D'$, we get a right-angled triangle with angle $\pi/10$ at A, hypotenuse b and opposite side $\frac{1}{2}(a-b)$. So

$$\sin(\pi/10) = \frac{\frac{1}{2}(a-b)}{b} = \frac{1}{2}(\phi - 1): \qquad \phi = 1 + 2\sin(\pi/10). \qquad \textbf{[2]}$$

[Seen – lectures and problems]

Q4. *The Fundamental Theorem of Arithmetic, FTA.*
Theorem (FTA). Every integer $n \geq 2$ can be written uniquely (to within order) as a product of prime factors. **[1]**
Proof. Existence. Induction. True for $n = 2$. Assume true for every integer $< n$. If n is not prime (i.e. is composite), it has a non-trivial divisor d $(1 < d < n)$. So $n = cd$ $(1 < c < n)$. So each of c, d is a product of primes, by the inductive hypothesis. So n is too, completing the induction. **[1]**
Uniqueness. Induction. True for $n = 2$. Assume true for every integer $< n$. If n is prime, the result holds, so assume n is composite. If it has two factorisations

$$n = p_1 \dots p_r = q_1 \dots q_s,$$

to show $r = s$ and each p is some q. As p_1 is prime and divides the product $n = q_1 \dots q_s$, it must divide at least one factor (w.l.o.g., q_1): $p_1 | q_1$. Then $p_1 = q_1$ as both p_1, q_1 are prime. Cancel p_1:

$$n/p_1 = p_2 \dots p_r = q_2 \dots q_s.$$

As n is composite, $1 < n/p_1 < n$. Then the inductive hypothesis tells us that the two factorisations above of n/p_1 agree to within order: $r = s$, and $p_2, \dots, p_r$ are $q_2, \dots, q_r$ in some order, as required. // **[1]**
Historical Note.

We owe Mathematics as a subject to the ancient Greeks. Of the 13 books of Euclid's Elements (EUCLID of Alexandria, c. 300 BC), three (Books VI, IX and X) are on Number Theory. From the ordering of the material in Euclid, it is clear that the Greeks knew that they did not have a proper theory of irrationals (i.e. reals). Although they did not state FTA, it had been assumed that they "knew it really", but did not state it explicitly. This view is contradicted by Salomon BOCHNER (1899-1982) (*Collected Papers*, Vol. 4, AMS, 1992). According to Bochner, the Greeks did *not* know FTA, nor have a notational system adequate even to state it! **[3]**

L. E. DICKSON (1874-1954) (*History of the Theory of Numbers* Vols 1-3, 1919-23) does not address the question of the Greeks and FTA! **[1]**

The first clear statement and proof of FTA is in Gauss' thesis (C. F. GAUSS (1777-1855); *Disquisitiones Arithmeticae*, 1798, publ. 1801). So we must attribute the result to Gauss. This is a wonderful example of the excellent mathematical *taste* of the young man who went on to be the greatest mathematician of all time. **[3]**

[Seen in less detail in lectures. The above is actually taken from my Analytic Number Theory notes, Lecture 1.]

Q5. *The electromagnetic theory of light.*
James Clerk Maxwell (1831-1879) [Whittaker, Ch. VIII: Maxwell].
A treatise on electricity and magnetism, Vol. 1, 2, OUP, 1891/1998.
Maxwell's Equations. If E, H are the electric intensity and the magnetic field in ES (electrostatic) units, cE in EM (electromagnetic) units, Maxwell's equations (in a vacuum) are

$$\text{div } E = 0, \quad \text{curl } E = -c^{-1}\partial H/\partial t; \quad \text{div } H = 0, \quad \text{curl } H = c^{-1}\partial E/\partial t. \quad [\mathbf{2}]$$

Whitaker (p. 245) writes: 'In this memoir (of 1865) the physical importance of the operators curl and div first became evident. These operators had, however, occurred frequently in the writings of Stokes ...'. Applying curl (recall curl curl = grad div - ∇^2, $= -\nabla^2$ here):

$$-\nabla^2 E = \text{curl curl} E = -\frac{1}{c}\frac{\partial}{\partial t}(\text{curl} H) = -\frac{1}{c2}\frac{\partial^2 E}{\partial t^2} : \qquad \nabla^2 E = c^{-2}\partial^2 E/\partial t^2,$$

and similarly

$$\nabla^2 H = c^{-2}\partial^2 H/\partial t^2. \qquad [\mathbf{2}]$$

This is the *wave equation*, for propagation of E, H with velocity c, the ratio of EM to ES units. This was known experimentally (c. 3×10^{10} cm/sec., c. 186,000 miles/sec.) to be (approx.) the *speed of light*. Thus, *electromagnetic forces are propagated with the speed of light.* This suggested (correctly) to Maxwell that *light waves are electromagnetic.* Recall the modern *electromagnetic spectrum*: in increasing order of wavelength, ..., x-rays, ultra-violet, visible spectrum, infra-red, radio waves, ... **[2]**

To do the mathematics above, one needs:
(i) *The wave equation*, the prototypical hyperbolic (linear, 2nd order) PDE: J. d'Alembert (1717-1783) in 1746.
(ii) *Vector calculus: grad, div and curl*: the divergence theorem, or Gauss' theorem (1813); Green's theorem (George Green (1793-1841; *Essay on magenetism and electricity*, 1828); Stokes' theorem (the curl theorem), 1854. **[2]**

Maxwell's EM theory of light builds on the work of Faraday on electromagnetic induction; together these constitute the two greatest advances of 19th C. Physics. From them flow the essentials of our modern life: electric power; radio, television, etc. **[2]**
[Seen – lectures]

Q6. *The mathematics of perspective.*

Perspective in the Ancient World **[3]**

We know from the writings of classical authors, such as the Roman author Vitruvius, and from surviving wall- and vase-paintings, that some elements of perspective were known to the ancient Greeks. Agatharchos used it for stage sets in the late 5th C. BC. Theoretical studies of perspective were made by Anaxagoras (above) and Democritus (above).

W: Systematic attempts to evolve a system of perspective are usually considered to have begun around the fifth century BC. in the art of Ancient Greece, as part of a developing interest in illusionism allied to theatrical scenery and detailed within Aristotle's Poetics as 'skenographia'.

Perspective in the Renaissance. **[3]**

As we have seen, perspective was known (at least in part) in the ancient world, but was then lost.

Filippo Brunelleschi (1377-1446) discovered the main principle of perspective – the use of vanishing points – and convinced his fellow-artists of this in a famous experiament of 1420 involving the chapel outside Florence Cathedral.

Leon Battista Alberti (1404-72), *Della pictura* (1435, printed 1511) gave the first written account of perspective.

Piero della Francesca (1410-92), *De prospectivo pingendi* (c. 1478). In his book, and in his painting, Piero della Francesca did much to popularise perspective, which spread throughout the Western art world.

Leonardo da Vinci (1452-1519); *Trattato della pittura.* Leonardo is usually regarded as the personification of Renaissance genius. He was a prolific inventor, an artist who wrote on perspective, and a mathematician.

Albrecht Dürer (1471-1528) of Nuremburg; *Investigations of the measurement with circles and straight lines of plane and solid figures* (1525-1538, German and Latin). Like Leonardo, Dürer was both a mathematician and an artist. He adopted perspective after visiting Italy.

Girard Desargues (1591-1661) and *Projective Geometry.* **[3]**

Desargues' first important book was *La Perspective* (1636). This led him on to his introduction of projective geometry in his *Brouillon projet (d'une atteinte aux evenements des rencontres d'une cone avec un plan)* (1639) – *Rough draft (of an attempt to deal with the outcome of a meeting of a cone with a plane).* Projective geometry is the mathematics of perspective (vanishing point = 'point at infinity').

[Seen – lectures]

Q7. *The Prime Number Theorem (PNT).*

PNT states that

$$\pi(x) := \sum\nolimits_{p \le x} 1 \sum li(x) := \int_2^x dt/\log t \sim x/\log x \qquad (x \to \infty) \quad (PNT)$$

This was conjectured on numerical grounds by GAUSS (c. 1799; letter of 1848) and A. M. LEGENDRE (1752-1833; in 1798, *Essai sur la Théorie des Nombres*). **[2]**

PNT was proved independently in 1896 by J. HADAMARD (1865-1963, French) and Ch. de la Vallée Poussin (1866-1962, Belgian). Both used Complex Analysis and ζ. **[2]**

In 1737 L. EULER (1707-1783) found his Euler product, linking the primes to $\sum_{n=1}^{\infty} 1/n^{\sigma}$ for real σ (later the Riemann zeta function). **[1]**

In 1859 B. RIEMANN (1626-66) studied

$$\zeta(s) := \sum\nolimits_{n=1}^{\infty} 1/n^s \qquad (s \in \mathbb{C})$$

using Complex Analysis (M2PM3), then still fairly new, developed by A. L. CAUCHY (1789-1857), 1825-29. He showed the critical relevance of the *zeros of* $\zeta(s)$ to the *distribution of primes.* One can show that:
(i) ζ can be continued analytically from $Re\ s > 1$ to the whole complex plane $\mathbb{C}$, where it is holomorphic except for a simple pole at 1 of residue 1 (III.3);
(ii) The only zeros of ζ outside the *critical strip* $0 < \sigma = Re\ s < 1$ are the so-called *trivial zeros* $-2, -4, \ldots, -2n, \ldots$ (trivial in that they follow from the *functional equation* for ζ;
(iii) PNT is closely linked to non-vanishing of ζ on the 1-line: $\zeta(1 + it) \neq 0$.

The *Riemann Hypothesis* (RH) of 1859 is that the only zeros of ζ in the critical strip are on the *critical line* $\sigma = \frac{1}{2}$. RH is still open, and is the most famous and important open question in Mathematics. Its resolution would have vast consequences for prime-number theory (especially error terms in PNT). **[4]**

Since counting primes relates to $\mathbb{N}$ ($\subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$), it seemed strange and unaesthetic to use complex methods. Great efforts were make to provide an *elementary proof*, over half a century.This was done in 1948 by A. SELBERG (1917-2009) and P. ERDÖS (1913-1996) (published independently in 1949). **[1]**

[Seen in lectures, in less detail. The above is taken from my Analytic Number Theory notes, Lecture 15.]

Q8. *The heliocentric theory.*
Aristarchus (of Samos, fl. c. 280 BC)
T. L. HEATH: *Aristarchus of Samos: The ancient Copernicus* (1913; 1981).

By observations made of eclipses, etc., Aristarchus was able to estimate the relative sizes of the earth, sun and moon. His estimates, though highly inaccurate by modern standards, were a good deal better than previous ones. He published his results in a book, 'On the dimensions and distances of the sun and moon'.

Both Archimedes (Dreyer, 138-8) and Plutarch (Dreyer, 138-140) gave detailed accounts of Aristarchus' views on the universe (the modern phrase 'solar system' is perhaps too coloured by hindsight here), in which they assert that Aristarchus pictured the earth as rotating about the sun – the *heliocentric* system of today. As for his book, Dreyer (p. 136) says flatly 'This treatise does not contain the slightest allusion to any hypothesis on the planetary system ...'; Boyer asserts (p. 180) that the book takes a geocentric view. But Heath, in his book on Aristarchus (p. iv) states that:
'... there is still no reason to doubt the unanimous verdict of antiquity that Aristarchus was the real originator of the Copernican hypothesis'.

Thus Aristarchus is a figure of tremendous importance; he has claims to be regarded as 'the ancient Copernicus'. **[3]**

Nicholas Copernicus (1473-1543) of Thorn (Niklas Koppernigk of Torun);
De revolutionibus orbium coelestium, 1543.

This work revolutionised astronomy by expounding the *heliocentric theory* – that the earth and other planets revolve around the sun. With Copernicus, the modern period of astronomy begins. See Dreyer Ch. XIII for a detailed account of this epoch-making achievement (and Dreyer Ch. VI and Week 2 for a discussion of Aristarchus and the heliocentric theory). **[4]**

Galileo Galilei (1564-1642); *Astronomy.*

Galileo invented (as well as an air thermometer) a telescope. With this, he began observations in 1609, observing
(i) the Mountains of the Moon; (ii) the four Moons of Jupiter;
(iii) the phases of Venus (incompatible with the geocentric system, since this shows that Venus orbits round the Sun).
This provided crucial observational support for the Copernican theory.
The Two Chief Systems (1632). Written in the form of a dialogue between three characters, this book supported the Copernican heliocentric theory, and brought Galileo into conflict with the Inquisition. **[3]** [Seen – lectures]

Q9. *The Archimedean solids.*

The five Platonic solids have regular polygonal faces *all of the same kind*. If several kinds of face are allowed, thirteen more 'semi-regular' solids are possible, as well as two infinite families, the 'prisms' and 'antiprisms'. We know from Pappus that Archimedes found the complete list (according to Heron, Archimedes ascribed one, the cuboctahedron, to Plato). **[4]**

The complete list is as follows:

1. Truncated tetrahedron, 3.6^2 (one triangle, two hexagons at each vertex)
2. Cuboctahedron, $(3.4)^2$ or 3.4.3.4 (triangle, square, triangle, square)
3. Truncated cube, 3.8^2
4. Truncated octahedron, 4.6^2
5. Small rhombicuboctahedron, 3.4^3
6. Great rhombicuboctahedron = truncated cuboctahedron, 4.6.8
7. Snub cube, $3^4.4$ (laevo and dextro forms)
8. Icosidodecahedron, $(3.5)^2$ (Coxeter, p.19)
9. Truncated dodecahedron, 3.10^2
10. Truncated icosahedron, 5.6^2 ('socer ball')
11. Small rhombicosidodecahedron, 3.4.5.4
12. Great rhombicosidodedcahedron = truncated icosidodecahedron, 4.6.10
13. Snub dodecahedron, $3^4.5$ (dextro and laevo)

See Heath I, Ch. IX, Plato, 294-5, II, Ch. XIII, Archimedes, 98-101. [The complete list need not be learned, but you should know that it exists and contains 13 members, including truncated forms of the five Platonic solids.] **[3]**

Carbon 60

H. W. (later Sir Harry) Kroto and others discovered a new form of carbon, C_{60}, in 1985 (Kroto, R. Curl and R. Smalley were awarded the 1996 Nobel Prize in Chemistry for this). The C_{60} molecule has carbon atoms at the 60 vertices of a truncated icosahedron. The shape is familiar to those who watch football: typically, soccer balls have white hexagonal and black pentagonal faces. This form of carbon (in addition to diamond and graphite) is called fullerene, from the 'geodesic dome' of the architect Buckminster Fuller. It is found in outer space (it has even been suggested that life on earth originated from this source). It has remarkable properties. **[3]**

[Seen – lectures]

Q10. *Centres of learning.*
Ancient academies

The earliest important centres of learning are from the ancient Greek world: the *Pythagorean school* at Croton ('Italian instep'), 6th C. BC; *Plato's academy* at Athens (c. 400 BC; Emperor Justinian closed the Academy at Athens 529 AD), and the *Museum of Alexandria* (from its foundation by Alexander the Great, c. 330 BC). These were succeeded by the *House of Wisdom* in Baghdad (Caliph al-Mamun, 809-833) (Baghdad, then under the Abbasid Caliphate, fell to the Mongols in 1258). **[2]**

Spain under the Moors was an important centre of learning. Here scholars from Arab, Jewish and Christian backgrounds lived together in Cordoba, Toledo and elsewhere; important work was done on translation. **[1]**

Early universities

The academies (Pythagorean, Athenian, Alexandrian) of the ancient world played the role of universities in their day, as did their Arab counterparts in Baghdad, Cordoba etc. By the 12th C., the modern concept of a university as an autonomous academic institution awarding degrees, and as a centre of learning, teaching and research, began to emerge. This was a gradual process. The earliest continental universities are Bologna (founded 1088; Royal Charter 1130), Salamanca (founded 1134, from a Cathedral School, 1130; Royal Charter 1218) and Paris (mid-12th C.). In Britain, Oxford and Cambridge simply describe themselves as 'founded in the 12th C.' and 'founded in the 13th C.' respectively. Scotland has four ancient universities: St. Andrews, 1410; Glasgow, 1451; Aberdeen, 1495; Edinburgh, 1583. **[2]**

Modern academies

The *Royal Society of London (RS)* was founded in 1660 (Charter 1663).

The French *Académie des sciences* was founded in 1666.

The Royal Prussian Academy of Sciences was founded in Berlin in 1700.

The St. Petersburg Academy was founded by Catherine I, Peter the Great's widow, in 1725. **[2]**

Modern universities

In the UK, the major civic universities were founded around 1900 (Birmingham, Manchester etc.). Universities did not all admit women as full members; public funding was so scarce that private means were often necessary to study at university, etc. This gradually improved. The next wave of new universities (York, Sussex etc.) came in the 1960s. The former polytechnics attained university status in 1992. **[3]**

[Seen in lectures, except for the last part – general knowledge]

Section B: answer 2 questions out of 4; 25 marks each.

Q1 *History of pi.*

The number pi (π) is defined as the ratio of the circumference of a circle to its diameter. [This tacitly assumes that this ratio is the same for all circles, but this belongs to the mathematics of the 19th and 20th C.: invariance under group actions, Haar measure etc.]

Egypt.

The Rhind [= Ahmed] Papyrus (c. 1,650 BC) contains (Problem 50: the equivalent of) $\pi \sim 4(8/9)^2 = 256/81 = 3\frac{13}{81} \sim 3\frac{1}{6}$.

Mesopotamia.

Susa tablets (found 1936): $\pi \sim 3:7:30 = 3 + \frac{7}{60} + \frac{30}{60^2} = 3\frac{15}{120} = 3\frac{1}{8}$. **[2]**

Bible, I Kings 7:23 (building of Solomon's Temple): $\pi \sim 3$. **[1]**

Archimedes (of Syracuse, c.287-212 BC): $3\frac{10}{71} < \pi < 3\frac{10}{70}$; thus $\pi \sim 22/7$, the 'Archimedean approximation'. **[2]**

Ptolemy (of Alexandria, fl. c. 127-150 AD). In the *Almagest*, Ptolemy obtains

$$\pi \sim 3:8:30 = 3 + \frac{8}{60} + \frac{30}{60^2} = \frac{377}{120} = 3.14166666....$$

[This value may have been obtained earlier by Apollonius of Perga (c. 262 - c. 190 BC), according to Heath (Vol. I, 232-235).] **[2]**

China. Liu Hui, 3rd C. AD: $\pi \sim 3.14$ (regular 96-gon); $\pi \sim 3.14159$ (regular $3,072$-gon).

Tsu Chung-chin (450-501 AD): $\pi \sim 22/7$; $\pi \sim 355/113$;

$$3.1415926 < \pi < 3.1415927.$$

[2]

Arabs. Al-Kashi (d. c. 1436), astronomer at Samarkand, and last of the major Arab figures: popularised decimal (rather than sexagesimal) fractions, and found π (actually, 2π) to 16 places:

$$2\pi = 6.28318\ 53071\ 79586\ 5.$$

[Various mnemonic devices for remembering the decimal expansion of π were given, to even greater accuracy.] **[2]**

Ludolph van Ceulen (1540-1610): found π to 20 places in 1596, and later to 35 places. Hence π was often called the *Ludolphine constant.* **[1]**

Francois Viète (1540-1603): first exact expression for π (*Variorum*, 1593): infinite product,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots . \qquad [\mathbf{2}]$$

He also calculated π to 10 sig. figs., unaware of previous better approximations.

John Wallis (1616-1703): Wallis' product for π (*Arithmetica infinitorum*, 1656):

$$\frac{2^2}{3^2} \cdot \frac{4^2}{5^2} \cdot \ldots \cdot \frac{(2m-2)^2}{(2m-1)^2} \to \frac{\pi}{2} \qquad (m \to \infty). \qquad [\mathbf{2}]$$

James Gregory (1638-1675): Gregory's series for *arctan* in 1668; $x = 1$ gives *Gregory's series* for π,

$$\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots \qquad [\mathbf{1}]$$

William Brouncker PRS (1620-1684): continued fraction for π (1655, unpublished; referred to by Wallis in 1656):

$$4/\pi = 1 + \frac{1^2}{2+}\ \frac{3^2}{2+}\ \frac{5^2}{2+} \cdots = 1 + \mathbf{K}_{n=1}^{\infty}\Big(\frac{(2n-1)^2}{2}\Big). \qquad [\mathbf{2}]$$

So π is *computable* (in the language of Turing – 20th C.): one could write a programme to calculate it, from Viète's product, Wallis' product, Gregory's series or Brouncker's continued fraction.

J. H. Lambert (1728-1777) in 1761: π is irrational. **[2]**

Ferdinand Lindemann (1852-1939) in 1882: π is transcendental.

Corollary: Impossibility of squaring the circle. For, constructible numbers are algebraic; if the circle could be squared, $\sqrt{\pi}$, hence also π, would be algebraic.

This resolves the famous problem first studied by Anaxagoras (d. 428 BC). **[2]**

C. F. Gauss (1777-1855), c. 1800, publ. posth.: Gauss gave an iteration related to the AM-GM inequality. Given $a, b > 0$, the iteration converges very rapidly to an 'arithmetico-geometric mean' $M(a, b)$ of a, b. This with modern computers has led to calculation of π to millions of decimal places: J. and P. BORWEIN, *Pi and the AGM*, 1987. **[2]**

[Seen, lectures. Full details – names, dates, values of approximations etc. – not expected.]

Q2 *Conics and orbits: Kepler and Newton.*
(i) *Polar equation of a conic.*

With focus F, directrix L distant ℓ (the latus rectum) from P, the focus-directrix property $PF = e.PL$ (Pappus, c. 290 AD) is $r = \ell - er\cos\theta$:

$$r(1 + e\cos\theta) = \ell, \qquad \frac{1}{r} = \frac{1}{\ell}(1 + e\cos\theta). \qquad [\mathbf{4}]$$

(ii) We need the components of velocity and acceleration along and perpendicular to the radius vector OP in polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta;$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}, \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta};$$

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\cos\theta(\dot{\theta})^2 - r\sin\theta\ddot{\theta},$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\sin\theta(\dot{\theta})^2 + r\cos\theta\ddot{\theta}.$$

So the components of velocity along and perpendicular to OP are $\dot{x}\cos\theta + \dot{y}\sin\theta = \dot{r}$ and $-\dot{x}\sin\theta + \dot{y}\cos\theta = r\dot{\theta}$ (both obvious). The components of acceleration are:

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - r(\dot{\theta})^2 \quad \text{along } OP, \qquad [\mathbf{2}]$$

$$-\ddot{x}\sin\theta + \ddot{y}\cos\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \quad \text{perpendicular to } OP \qquad [\mathbf{2}]$$

(no $\ddot{r}$ – 1st – term; 2nd term $2\dot{r}\dot{\theta}$ $(\cos^2 + \sin^2 = 1)$; no $r(\dot{\theta})^2$ – 3rd – term; 4th term $r\ddot{\theta}$ $(\cos^2 + \sin^2 = 1)$).
(iii) If the force is *central* (along OP), then there is no acceleration perpendicular to OP. So $h := r^2\dot{\theta}$ is constant (the angular momentum per unit mass). But if A is the area swept out by the radius vector, $dA = \frac{1}{2}r.rd\theta$: $\dot{A} = \frac{1}{2}r^2\dot{\theta}$. So

$$\dot{A} = \frac{1}{2}h = constant:$$

the radius vector sweeps out equal areas in equal times (Kepler's Second Law – equivalent to central forces). [**4**]

For a central force: write $u := 1/r$. So by (ii), $\dot{\theta} = h/r^2 = hu^2$,

$$\frac{dr}{dt} = \frac{d}{dt}(1/u) = \frac{d}{du}(1/u)\frac{du}{dt} = -\frac{1}{u^2}.\frac{du}{d\theta}.\frac{d\theta}{dt} = -\frac{1}{u^2}.\frac{du}{d\theta}.hu^2: \quad \dot{r} = -hdu/d\theta.$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}(-h\frac{du}{d\theta}) = -h\frac{d}{d\theta}(\frac{du}{d\theta}).\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} : \quad \ddot{r} = -h^2u^2d^2u/d\theta^2.$$

So by (ii), with a the acceleration a along OP towards O,

$$-a = \ddot{r} - r(\dot{\theta})^2 = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}.h^2u^4 : \qquad \frac{d^2u}{d\theta^2} + u = \frac{a}{h^2u^2}. \qquad (*)$$

Newton's Inverse Square Law of Gravity is that the acceleration $a = c/r^2 = cu^2$ for c constant: $c = a/u^2$ is constant. So by $(*)$, this is

$$\frac{d^2u}{d\theta^2} + u = \frac{c}{h^2} = b, \qquad (DE) \quad \mathbf{[4]}$$

say.

(iv) The differential equation (DE) has general solution $u = b + c_1\cos\theta + c_2\sin\theta$. We may choose the initial line $\theta = 0$ to make $du/d\theta = 0$ there; then $c_2 = 0$, and $u = b + c_1\cos\theta$:

$$1/r = b + c\cos\theta.$$

But the polar equation of a conic of eccentricity e and latus rectum ℓ is

$$r(1 + e\cos\theta) = \ell : \qquad \frac{1}{r} = \frac{1}{\ell}(1 + e\cos\theta).$$

This identifies the path of a particle moving under the inverse square law as a *conic*. // **[4]**

(v) This is the most important single result in Newton's *Principia* (1687), itself the most important book in the history of science. Our orbit round the Sun is closed, so being a conic, it is an *ellipse*. The eccentricity of the ellipse gives us our seasons.

Johannes Kepler (1571-1630): Assistant to Tycho Brahe (1546-1601) at Prague Observatory from 1600; succeeded him in 1601; *Astronomia Nova*, 1609, Kepler's first and second laws (*Harmonices mundi*, 1619, Kepler's third law). Brahe spent twenty years observing planetary orbits; Kepler spent twenty years analysing his data.

Newton's derivation was geometrical. He avoided calculus (his 'method of fluxions'), preferring to solve an old problem by established methods.

The geometrical content, as noted, is Pappus' focus-directrix property of conics (Alexandria, c. 290 AD), used via polar coordinates (cartesian geometry, 17th C.).

The differential equations method above (technically easier for a modern audience) belongs to the 18th C. (the Bernoullis, Euler etc.). **[5]**

Seen – lectures + problem sheets.

Q3 *Normal distribution; Central Limit Theorem; Least Squares.*
Abraham de Moivre (1667-1754)
The Doctrine of Chances (DC; 1718; 2nd ed. 1738; 3rd ed. 1756, posth.)

De Moivre was born in France, a Huguenot (= Protestant). He fled to England to escape religious persecution after the revocation of the Edict of Nantes in 1685 by Louis XIV. He was elected FRS in 1697.

In 1733 de Moivre derived the normal curve ("error curve": the term "normal" came later)

$$\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$$

in a 7-page note in Latin *Approximatio ...* (circulated, but not published). An English translation of the *Approximatio* was incorporated in the 2nd and 3rd editions of DC. De Moivre showed that in n Bernoulli trials with parameter p

$$P((S_n/n - p)\sqrt{n}/\sqrt{pq} \in [a, b]) \to \int_a^b \phi(x)dx \qquad (n \to \infty)$$

(recall that $P(S_n = k) = \binom{n}{k} p^k q^{n-k}$, so the LHS is this summed over k with $np + a\sqrt{npq} \leq k \leq np + b\sqrt{npq}$). This is the special case for Bernoulli trials of the "Law of Errors", or *Central Limit Theorem* (CLT). Note that this refines Bernoulli's theorem (*Ars conjectandi*, 1713) by telling us *how fast* S_n/n tends to p: at rate $1/\sqrt{n}$, with a normal limit: in modern terminology,

$$((S_n/n) - p)\sqrt{n}/\sqrt{pq} \sim N(0, 1). \qquad [\mathbf{6}]$$

One can see that this result will need an estimate for the large factorials occurring in $\binom{n}{k} = n!/(k!(n-k)!)$, as in Stirling's formula, 1730:

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}} \qquad (n \to \infty). \qquad [\mathbf{1}]$$

Adrien-Marie Legendre (1725-1833), Professor at the Ecole Militaire
Nouvelles méthodes pour la détermination des orbites des comètes (1805) (supplement 1806, 2nd suppl. 1820).

The orbits of planets and comets, being ellipses and so conics, are determined by two parameters. For flexibility in choice of frame of reference, it may be useful to use p parameters, $\theta_1, \ldots, \theta_p$, where p is small. With exact measurements, these could be determined by p observations, $y_1, \ldots, y_p$ say, but in practice the y_i are polluted by measurement error. However, we may take n readings where n is large (the larger the better – much larger than p). The usual set-up is

$$y_i = \sum_{j=i}^{p} x_{ij}\theta_j + \epsilon_i \qquad (i = 1, \ldots, n),$$

where the x_{ij} are known, the ϵ_i are independent errors of measurement, and the θ_j are to be estimated. How should one estimate the θ_j? Legendre suggested the *method of least squares*: minimise the sum of squares

$$SS := \sum_i (y_i - \sum_j x_{ij}\epsilon_j)^2 = \sum (\text{observed - expected})^2.$$

The p conditions $\partial SS/\partial\theta_j = 0$ give p simultaneous linear equations in p unknowns, the *normal equations* NE; the solutions $\hat{\theta}_j$ give the *least-squares estimators* for the parameters. The method of least squares, in Statistics, is of enormous practical importance. **[6]**

Carl Friedrich Gauss (1777-1855)

Gauss published his work, on the orbit of Ceres (1801), Pallas (another planetoid) and least squares in
Theoria motus corporum coelestium in sectionibus conicus solem ambientem (1809) (TM).

Legendre's book of 1805 presented and named the method, but did not link it to Probability Theory. Gauss, however, *linked least squares with the normal law.* He said in TM 'On the other hand, our principle (principium nostrum), which we have made use of since the year 1795, has lately been published by Legendre ...'. No doubt Gauss was telling the literal truth. Understandably, however, Legendre felt slighted that Gauss should thus seek to claim priority for something first published by Legendre, and protested, first in private correspondence, and in 1820 publicly in a Second Supplement to his 1805 book. **[6]**

Pierre-Simon de Laplace (1749-1827), Professor at the Ecole Normale and the Ecole Polytéchnique.

Théorie Analytique des Probabilités (1812) (TAP) (2nd ed. 1814, 3rd 1820). Laplace gave (Ch. IV of TAP) a thorough treatment of the *method of least squares* (due to Legendre and Gauss). He brought together the roles of the normal distribution in the Central Limit Theorem and the Method of Least Squares, thus achieving the *Gauss-Laplace synthesis.* **[6]**

[Seen – lectures.]

Q4 *Solution of polynomial equations.*
Egypt: Linear equations are found in the Rhind (or Ahmes) Papyrus, c. 1650 BC. **[1]**
Mesopotamia: Some quadratic equations were studied c. 2000 BC (also some cubics). **[1]**
Diophantus of Alexandria, c. 250 AD: For positive rational solutions, Diophantus dealt with quadratic equations completely, and some cubics. **[1]**
India: Brahnagupta (c. 628 AD): General solution of quadratics. **[1]**
al-Khwarizmi (d. 850), *Al-Jabr wa al-Muqabala*. **[1]**

This book synthesised the Mesopotamian, Greek and Hindu algebra traditions, and was the first to do so. **[1]**
Omar Khayyam (c. 1100): Algebra – quadratics and cubics. **[1]**

Thus pre-modern work on the solution of polynomial equations was restricted to real (and sometimes, to positive) roots. Within this framework, it is impossible to account for the fact that a polynomial of degree n (recall that $n \leq 3$ here) may not have n real (or positive) roots).

Turning to the modern European period:
Geronimo Cardano (1501-1570); *Ars Magna*, 1545.

Cardano's *solution of the cubic* (published here in 1545) marks 'the beginning of the modern period in mathematics'. Scipione del Ferro (c. 1465-1526), Professor of Mathematics at Bologna, solved the cubic but did not publish his results. Niccolo Tartaglia (c. 1500-1557) (b. Niccolo Fontana; Tartaglia = stammerer), knowing of del Ferro's solution, found one himself, by 1541. Predictably, this led to a priority dispute with Cardano.
Complex numbers.

Cardano, in Ch. 37 of *Ars Magna*, solves

$$x(10 - x) = 40,$$

obtaining roots $5 \pm \sqrt{-15}$, and notes that

$$(5 + \sqrt{-15})(5 - \sqrt{-15}) = 25 - (-15) = 40.$$

Thus 'Without having fully overcome their difficulties with irrational and negative numbers, the Europeans added to their problems by blundering into what we now call complex numbers.' Though complex numbers were not properly assimilated into mathematics till much later, they enter the stage with *Ars Magna*.

Also: 'whenever the three roots of a cubic are real and non-zero, the

Cardan-Tartaglia formula leads inevitably to square roots of negative numbers'. **[3]**
Carl Friedrich Gauss (1777-1855)
Fundamental Theorem of Algebra (1799)

Gauss' doctoral thesis (in Latin: 'A new proof that every polynomial of one variable can be factored into real factors of the first or second degree') was published in 1799.

Despite its name, this result is a theorem of *analysis*, not of *algebra*. Its proof was less rigorous than Gauss' usual standard: he assumed properties of continuous functions later proved by Bolzano. **[3]**
Augustin-Louis Cauchy (1789-1857), Professor at the Ecole Polytéchnique and later the Sorbonne.

Cauchy's *Cours d'analyse* (Ecole Polytéchnique, 1821) contains a proof of the Fundamental Theorem of Algebra: every complex polynomial of degree n has n complex roots (counted according to multiplicity). **[3]**

The modern proof uses *Liouville's theorem* (Joseph Liouville (1809-1882), lectures in 1847 – actually published by Cauchy in 1844): an entire (i.e. holomorphic throughout the complex plane $\mathbb{C}$) bounded function is constant. Thus it took over twenty years before the full power of Cauchy's new subject of Complex Analysis was properly brought to bear on the Fundamental Theorem of Algebra – incidentally, revealing in so doing that the result, being a theorem in Analysis, is a misnomer. **[3]**
N. H. Abel (1802-29): Insolubility of the quintic (1829).

Although the quintic has five roots, by the Fundamental Theorem of Algebra above, Abel showed that – in contrast to polynomials of degree up to four, which can be solved, as Cardano showed – quintics are *not soluble by radicals*: there can be *no* formula/algorithm/method for expressing the roots in terms of the coefficients. Similarly for polynomials of higher degree. So there is a fundamental split: polynomial equations are soluble by radicals for degree up to 4, but not for degree 5 or higher. **[3]**
Evariste Galois (1811-1832).

Galois was the first to study *field extensions* systematically. This is the key to the *algebraic closure* of the complex plane $\mathbb{C}$, but not of the real line $\mathbb{R}$: a polynomial of degree n with coefficients in $\mathbb{C}$ has n roots in $\mathbb{C}$, but one with coefficients in $\mathbb{R}$ need not have n roots in R. **[3]**
[Seen – lectures]

N. H. Bingham