m3hprob6.tex

M3H PROBLEMS 6. 20.2.2018

Q1 (*Viète's infinite product for* π: Francois Viète (1540-1603) in 1593).
(i) Show that if A_n is the area of the regular inscribed n-gon to a unit circle, then

$$A_n/A_{2n} = \cos(\pi/n).$$

(ii) Deduce Viète's infinite product for π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}.\frac{\sqrt{2+\sqrt{2}}}{2}.\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}.\cdots.$$

(Viète, *Variorum* liber VIII, 1593: the first exact expression for π).

Q2 (*Girard's formula for spherical excess*: Albert Girard (1595-1632), *Invention nouvelle en algèbre*, 1629).

Show that on sphere of radius r, a spherical triangle wtih angles A, B, C has area

$$S = r^2(A + B + C - \pi).$$

Q3 (*Wallis' product for* π: John Wallis (1616-1703), *Arithmetica infinitorum*, 1656).

By integrating by parts, or otherwise, show that if $I_n := \int \sin^n x \, dx$,

$$nI_n = -\sin^{n-1} x \cos\ x + (n-1)I_{n-2}.$$

Deduce that if $J_n := \int_0^{\pi/2} \sin^n x \, dx$, one has the *reduction formula*

$$J_n = \frac{n-1}{n}.J_{n-2}.$$

Hence show that

$$J_{2m} = \frac{2m-1}{2m}.\frac{2m-3}{2m-2}.\cdots.\frac{1}{2}.\frac{\pi}{2};$$

$$J_{2m+1} = \frac{2m}{2m+1}.\frac{2m-2}{2m-1}.\cdots.\frac{2}{3}.$$

Show that $J_{2m+1} \le J_{2m} \le J_{2m-1}$, and hence that (Wallis' product)

$$\frac{2^2}{3^2}.\frac{4^2}{5^2}.\cdots.\frac{(2m-2)^2}{(2m-1)^2} \to \frac{\pi}{2} \qquad (m \to \infty).$$

NHB