m3hsoln1.tex

M3H SOLUTIONS 1. 23.1.2018

Q1 (Theorem of Thales). Let the triangle be ABC, with AB a diameter of a circle, with centre O and radius r say. Join CO. Triangle ΔAOC is isosceles ($AO = CO = r$), so with θ the angle $\angle CAO$, $\theta = \angle ACO$ also. So $\angle AOC = \pi - 2\theta$ (to make the angle sum of ΔAOC π). So the complementary angle $\angle BOC = 2\theta$. But ΔBOC is also isosceles, so $\angle OCB = \angle OBC = \frac{1}{2}\pi - \theta$ (to make the angle sum of ΔOCB π). So $\angle ACB = \angle ACO + \angle OCB = \theta + (\frac{1}{2}\pi - \theta) = \frac{1}{2}\pi$. //

Q2 (Theorem of Pythagoras).
First proof (Draw a diagram). Let ΔABC be right-angled, with right angle at A, hypotenuse $BC = \ell$ and other sides $AC = \ell_1$ and $AB = \ell_2$. With D the foot of the perpendicular from A to BC, the area b of ΔABC is the sum of the areas b_1, b_2 of $\Delta ACD, \Delta ABD$, namely $b(\ell_1/\ell)^2$, $b(\ell_2/\ell)^2$:

$$b = b((\ell_1/\ell)^2 + (\ell_2/\ell)^2): \qquad \ell_1^2 + \ell_2^2 = \ell^2.$$

This is Pythagoras' theorem. //

The same 'similarity and scaling' argument applies to each of the three 'triangle + square' figures, just as to the triangles. It gives the same conclusion, but more in the Greek style of geometry, as now ℓ^2 is interpreted as the area of the square on the hypotenuse, etc.
Second proof (Draw a diagram). (i) Using the notation of the first proof, draw a square of side $\ell_1 + \ell_2$, with vertices $P_1, \ldots, P_4$ (with P_1P_2 horizontal, say). Mark off points $Q_1, \ldots, Q_4$ with $P_1Q_1 = \ldots = P_4Q_4 = \ell_1$ (with each Q_i to the right of P_i, say). The Q_i form the vertices of a square, with side ℓ.
(ii) Draw the vertical through Q_1, meeting P_3P_4 in R_1, and the horizontal through Q_4, meeting P_2P_3 in R_4, say; let Q_4R_4 and Q_1R_1 meet in S. The four right-angled triangles $\Delta P_1Q_1Q_4$, $\Delta P_2Q_2Q_1$, $\Delta P_3Q_3Q_2$, $\Delta P_4Q_4Q_3$ are congruent to the triangles $\Delta P_1Q_1Q_4$, ΔQ_1SQ_4, $R_4P_3R_1$, ΔSR_4R_1. So their two area-sums are the same, a say. The square $Q_4SR_1P_4$ has area ℓ_1^2; square $Q_1P_2R_4S$ has area ℓ_2^2; square $Q_1Q_2Q_3Q_4$ has area ℓ^2.
(iii) The area of square $P_1P_2P_3P_4$ is $a + \ell^2$ in (i) and $a + \ell_1^2 + \ell_2^2$ in (ii). Equating these gives Pythagoras' theorem. //

Q3 (Angle-sum of a plane triangle: Draw a diagram). If the triangle is ΔABC, draw through C the line L parallel to AB. If $\theta := \angle CAB$, θ is also the angle between AC produced and L (as L and AB are parallel), and so also between AC and L on the same side of L as ΔABC) ($\angle LCA$ is called the *alternate angle* to $\angle CAB$). Similarly, $\phi := \angle ABC = \angle LCB$, again by alternate angles. But at C, $\angle LCA + \angle ACB + \angle BCL = \pi$, i.e. $\theta + \angle ACB + \phi = \pi$, i.e. $\angle CAB + \angle ACB + \angle ABC = \pi$. So ΔABC has angle-sum π. //

Q4 (Star pentagram and golden section: Euclid, Book 6 Prop. 30).
(i) $\Delta AD'B$ is isosceles ($AD' = BD'$ by symmetry), so $\angle D'AB = \angle D'BA, = \theta$ say. Write $\theta' := \angle EBD$. Then $2\theta + \theta' = 3\pi/5$ (at B, the interior angle is $3\pi/5$, as the complementary exterior angle is $2\pi/5$). Angle $\angle AD'B = \pi - 2\theta$ (angle-sum in $\Delta AD'B$). So the complementary angle $\angle AD'C' = 2\theta$. Triangle $\Delta AD'C'$ is isosceles, by symmetry; its angle-sum gives $4\theta + \theta' = \pi$. Eliminating θ', $\pi - 4\theta = 3\pi/5 - 2\theta$: $\theta = \pi/5$, and then $\theta' = \pi/5$. So the interior angles are *trisected*.
(ii) $\Delta AEB'$ is isosceles ($\angle EAB' = \pi/5$; $\angle AEB' = 2\pi/5$; so $\angle EB'A = 2\pi/5$. So $a = AE = AB'$. So if $b := AC'$, the sides of the inner pentagon are $a - b = C'B'$, etc.
(iii) Triangles ΔEAB and $\Delta EC'A$ are similar (both isosceles, with angles $\pi/5, \pi/5, 3\pi/5$), with sides $a, a, a+b$ and b, b, a. So

$$\phi := \frac{a}{b} = \frac{a+b}{a} = 1 + \frac{1}{\phi} : \qquad \phi^2 - \phi - 1 = 0 : \qquad \phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$$

(we take the + sign in $\pm$ since $\phi > 0$).
(iv) The outer and inner pentagons have sides a, $a - b$, whose ratio is

$$(a-b)/a = 1 - 1/\phi = 2 - \phi = \frac{1}{2}(3 - \sqrt{5}).$$

(v) Dropping the perpendicular $C'C''$ from C' to AE, $\cos(\pi/5) = \frac{1}{2}a/b = \frac{1}{2}\phi$:

$$\phi = 2\cos(\pi/5).$$

Dropping the perpendicular AA'' from A to $C'D'$, we get a right-angled triangle with angle $\pi/10$ at A, hypotenuse b and opposite side $\frac{1}{2}(a-b)$. So

$$\sin(\pi/10) = \frac{\frac{1}{2}(a-b)}{b} = \frac{1}{2}(\phi - 1) : \qquad \phi = 1 + 2\sin(\pi/10).$$

NHB