m3hsoln2.tex

M3H SOLUTIONS 2. 30.1.2018

Q1 (Angle at centre twice angle at circumference).

Let the chord be AB,C be the point on the circumference, O the centre of the circle. Required $\angle AOB = 2\angle ACB$. Let $\theta := \angle OAC$, $\phi := \angle OBC$. Triangles ΔAOC, ΔBOC are isosceles (two sides are the radius, r say). So $\angle OCA = \theta$, $\angle OBA = \phi$. So AB subtends $\angle ACB = \theta + \phi$ at the circumference. In ΔAOC, $\angle AOC = \pi - 2\theta$ (angle sum is π), and similarly $\angle BOC = \pi - 2\phi$. The three angles are O sum to 2π; the two just mentioned sum to $2\pi - 2\theta - 2\phi$. So $\angle AOC = 2(\theta + \phi) = 2.\angle ACB$. //
Note that if the chord goes through the centre, the angle at the centre is π, so the angle at the circumference ('angle in a semi-circle') is π, and we recover the theorem of Thales.

Q2 (Angles in the same segment).

Both angles subtend the same angle at the centre, so by Q1 they are equal.

Q3 (Opposite angles of a cyclic quadrilateral sum to π).

If the opposite angles are $\theta := \angle ABC$, $\phi := \angle ADC$: by Q1, the arc ABC subtends angle 2θ at), and arc ADC subtends 2ϕ at O. But these angles sum to 2π (the total angle at O). So $\theta + \phi = \pi$. //

Q4 Schläfli symbols and Platonic solids).
(i) As in the star pentagram: as we go round the perimeter of a regular p-gon,the direction changes by $2\pi/p$ at each vertex. So the interior angle at each vertex is $\pi - 2\pi/p = \pi(1 - 2/p)$. But q of these can fit together in a polyhedron iff $q.\pi(1 - 2/p) < 2\pi$. So the requaired inequality is

$$q(1 - 2/p) < 2.$$

(ii) Tetrahedron: triangular faces, 3 meet at a vertex: $\{3, 3\}$.
Octahedron: triangular faces, 4 meet at a vertex: $\{3, 4\}$.
Cube: square faces, 3 to a vertex: $\{4, 3\}$.
Dodecahedron: pentagonal faces, 3 at a vertex: $\{5, 3\}$.
Icosahedron: triangular faces, 5 at a vertex: $\{3, 5\}$.

Q5.
Tetrahedron: $F = 4, V = 4, E = 6, F + V - E = 2$.

Octahedron: $F = 8, V = 6, E = 12, F + V - E = 2$.
Cube: $F = 6, V = 8, E = 12, F + V - E = 2$.
Dodecahedron: $F = 12, V = 20, E = 30, F + V - E = 2$.
Icosahedron: $F = 20, 12, E = 30, F + V - E = 2$.

Note. 1. That $F + V = E + 2$ holds for *all* polyhedra: *Euler's formula* (Week 7, L20). It is result on (combinatorial) *topology* (Weeks 9, 10).
2. There is a sense in which the octahedron and cube are *dual*, the dodecahedron and icosahedron are *dual*, and the tetrahedron is *self-dual*. This involves the ideas of *projective geometry* (Weeks 6 and 8).

Q6. **Theorem (Euclid)**. There are infinitely many primes.

Proof (direct). List the primes in order of size: $p_1 = 2, p_2 = 3, p_3 = 5$ etc. For each $n \in \mathbb{N}$, consider

$$N := 1 + p_1 p_2 \dots p_n.$$

Then p_1 does not divide N: N has remainder 1 when divided by p_1. Similarly, $p_2, \dots, p_n$ do not divide N. So, since N has a prime factor by FTA, either N is prime ($= p_{n+1}$), or there is a prime p_{n+1} between p_n and N. Either way, we can continue the list of primes, beyond the nth for each n. Keep going, 'ad infinitum'! So, there are infinitely many primes. //

A direct proof is usually preferred to a proof by contradiction: "inside every proof by contradiction there is a direct proof struggling to get out". See e.g. (a great book! – highly recommended)
George PÓLYA, *How to solve it: A new aspect of mathematical method*, Princeton, 1945, p/b, 1971 (Penguin, p/b, 1990).

The proof by contradiction is essentially the same but slightly shorter:

Proof (by contradiction). Assume not. Then some list $p_1, \dots, p_n$ exhausts the primes. Consider

$$N := 1 + p_1 p_2 \dots p_n.$$

Then p_1 does not divide N: N has remainder 1 when divided by p_1. Similarly, $p_2, \dots, p_n$ do not divide N. So as these are all the primes, N does not contain a prime factor, contradicting FTA. // NHB