

### M3H SOLUTIONS 3. 6.2.2018

Q1 *Archimedes' sphere-cylinder theorem.*

Consider the element  $dA$  of surface area of the sphere between angles  $\theta$  and  $\theta + d\theta$  ( $\theta$  is the angle between the line  $OP$  from the centre  $O$  to the point  $P$  on the sphere with the axis of the cylinder). To first order, this is a circular band of radius  $r \sin \theta$  and thickness  $r d\theta$  [as  $r \sin \theta$  is the radius of this circular band]. So

$$dA = 2\pi r^2 \sin \theta d\theta.$$

Integrate over  $\theta \in [\alpha, \beta]$ :

$$A = 2\pi r^2 \int_{\alpha}^{\beta} \sin \theta d\theta = 2\pi r^2 [\cos \alpha - \cos \beta].$$

But this is the area of the part of the cylinder between  $\theta = \alpha$  and  $\theta = \beta$  [the slice has height  $r \cos \beta - r \cos \alpha$ ].

Q2 *Conics.*

Take the axis  $L$  of the cone as the  $z$ -axis  $Oz$ . Then the radius  $r$  of the cone at height  $z$  is proportional to  $z$ . So the cone has equation of the form

$$x^2 + y^2 = k^2 z^2.$$

The plane  $\Pi$  of section has equation of the form

$$ax + by + cz = d.$$

If  $\Pi$  is horizontal (perpendicular to  $L$ ),  $c = 0$  and the locus of intersection is a circle (whose equation is of the second degree). If not, we can eliminate  $z$  between the two equations above. We are left with an equation of the second degree in  $x$  and  $y$ , as required. [Note that all types of conic can arise in this way, as expected.]

Q3. *Apollonius' 3-line problem.*

If the lines are  $L_i$ :  $x \cos \alpha_i + y \sin \alpha_i - c_i = 0$ , the distance  $d_i$  from  $P = (x, y)$  to  $L_i$  is  $PL_i = x \cos \alpha_i + y \sin \alpha_i - c_i$  (to within sign). So the locus of  $d_1^2 = cd_2d_3$  is

$$(x \cos \alpha_1 + y \sin \alpha_1 - c_1)^2 = \pm c(x \cos \alpha_2 + y \sin \alpha_2 - c_2)(x \cos \alpha_3 + y \sin \alpha_3 - c_3)$$

(the sign can be determined from one point) – a conic.

(b) *Apollonius' 4-line problem*. Similarly, the locus of  $d_1d_2 = cd_3d_4$  is

$$(x \cos \alpha_1 + y \sin \alpha_1 - c_1)(x \cos \alpha_2 + y \sin \alpha_2 - c_2) = \\ \pm c(x \cos \alpha_3 + y \sin \alpha_3 - c_3)(x \cos \alpha_4 + y \sin \alpha_4 - c_4),$$

again a conic.

Q4 *Focus-directrix property of conics: Pappus*.

Let  $F = (x_0, y_0)$  be the focus,  $L : x \cos \alpha + y \sin \alpha - c = 0$  be the directrix. If  $P = (x, y)$ , the locus of  $PF = e.PL$  is

$$(x - x_0)^2 + (y - y_0)^2 = e^2(x \cos \alpha + y \sin \alpha - c)^2,$$

a conic.

NHB