m3hw3.tex **Week 3 25-30.1.18**

THE GREEKS: ARCHIMEDES TO PAPPUS

Dramatis Personae
Eudemus (of Rhodes), fl. c. 320 BC: history of mathematics (now lost: W4)
Archimedes (of Syracuse, c. 287-212 BC)
Apollonius (of Perga, Asia Minor, c.262 – c.190 BC)
Aristarchus (of Samos, fl. c. 280 BC)
Eratosthenes (of Cyrene, 276-194 BC)
Hipparchus (of Nicea, c. 180 – c. 125 BC)
Menelaus (of Alexandria, c. 100 AD)
Ptolemy (of Alexandria, fl. c. 127-150 AD)
Heron (of Alexandria, 3rd C. AD)
Diophantus (of Alexandria, fl. c. 250 AD)
Pappus (of Alexandria, fl. c. 290 AD)
Sources
B Ch. 8-11, and other sources for Week 2;
E. J. DIJKSTERHUIS, *Archimedes*, 2nd ed. (with an addendum by W. R. Knorr), PUP 1987 (1st ed. 1938-44, Dutch, Eng. tr. 1957)
H. S. M. COXETER, *Regular polytopes*, 3rd ed., Dover, 1973 (1st ed. 1963).
H. M. CUNDY & A. P. ROLLETT, *Mathematical models*, 2nd ed., OUP, 1961.

Archimedes

Archimedes lived in Syracuse, SE Sicily, then part of the Hellenistic world. He may have studied at Alexandria, and was in contact with mathematicians there. Syracuse sided with Carthage in the Punic Wars, and fell to Rome in 212 BC. Archimedes played a leading part in the defence of Syracuse during the long siege 214-212 BC, inventing engines of war: catapults to hurl stones, cranes of pull ships from the water and drop them, etc. (the siege is described in details by Plutarch, Livy and Polybius). The Roman general Marcellus admired Archimedes and ordered his life to be spared. Despite this, he was (as legend has it) killed by a Roman soldier who interrupted him at work drawing diagrams in the sand. Cicero relates that, on coming to Sicily as quaestor in 75 BC, he found 'at the gate of Achradina' an overgrown grave bearing the figure of a sphere with a cylinder circumscribed – Archimedes chose this inscription for his grave, to commemorate his favourite theorem (below).

Archimedes is without doubt the great mathematician of the ancient world, just as Newton is for the early modern period and Gauss is for the later modern period. These are the greatest three mathematicians in history. It is impossible to make sensible comparisons between them, as their times were so completely different. We shall discuss each of them in full, in chronological order.

Mechanics

Archimedes constructed pulleys, by which he was able to move ships in Syracuse harbour single-handed (A variant of this is given on B p.140).[1]

Statics

Law of the lever (moments, fulcrum etc.): B 8.2.

Hydrostatics

Archimedes is known to the man in the street as having discovered Archimedes' Principle (that a floating body displaces its own weight of water), and for then (reputedly) having run down the street naked from his bath shouting 'Eureka' ('I have found it.'). He wrote a treatise 'On floating bodies', for which he has been called the 'father of mathematical physics' – and which is still of value for, e.g., ship design. Other contributions include the Archimedes (helical) screw, still used for irrigation, and (reputedly) testing precious metals for purity by immersing them in water.

The Sand-reckoner (B 8.5).

Perhaps because of the transition from Attic to Ionian numerals around this time, Archimedes did not scorn calculation. He contributed to the debate (involving Aristarchus) about heliocentric universe, constructing 'astronomically large' numbers for this purpose (and touching on the law of exponents).

Calculation of π (B 8.5).

By inscribing and circumscribing n-gons to a circle for large n, Archimedes obtained the estimate

$$3\frac{10}{71} < \pi < 3\frac{10}{70}$$

– superior to those of the Egyptians and Mesopotamians.

On Spirals

In this book, Archimedes used the 'Archimedean spiral', the curve $r = c\theta$ (in modern polar coordinates), to trisect the angle, calculate areas, etc.

Quadrature of the Parabola (B 8.7).

[1] Pulleys were a standard part of the Physics curriculum when I was young (GCSE, as it then was, 1960). *Mechanical advantage* and *velocity ratio* are the technical terms that I remember.

Here Archimedes used the method of exhaustion to calculate the area A_1 of a parabolic segment (region bounded by a parabola and line) as

$$A_1 = \frac{4}{3} A_2,$$

where A_2 is the area of the triangle with the same base and 'height'. This first quadrature of a conic had considerable impact at the time.

On Conics and Spheroids (B 8.8).

Here Archimedes found the area of an ellipse with semi-axes a, b (in particular the area of a circle of radius r) as respectively [2]

$$A = \pi ab, \qquad A = \pi r^2.$$

He also solved the corresponding 3-dimensional problems on volumes of segments of conoids.

On the Sphere and Cylinder. (B 8.9, 10).

Here Archimedes compares the volume of a sphere and circumscribing cylinder of the same radius r, and also their areas. Hence,

$$V = \frac{4}{3}\pi r^3, \qquad S = 4\pi r^2.$$

This led Archimedes to request that such a sphere and cylinder be inscribed on his grave.

The work (Proposition 2, Book II) also contains a volume calculation involving a cubic. Archimedes in essence here anticipated the general solution of the cubic equation some 1,750 years later.

The Method (B 8.13).

In 1899 J. L. Heiberg (Handout) found a palimpsest (a parchment which has been re-used) in the Library of the Monastery of the Holy Sepulchre in Jerusalem. (This probably originated in Constantinople; it was there that Heiberg examined it in 1906-8 – see Dijksterhuis 44, 36.) Beneath the mediaeval liturgical text, he was able to decipher a previously unknown text by Archimedes, *The Method of Mechanical Theorems for Eratosthenes* (briefly, *The Method*). Here Archimedes describes the method which led him to, e.g., his quadrature of the parabola: an application of his 'principle of the lever' via a balancing argument. This argument also led Archimedes to his favourite theorem on the sphere and cylinder.

[2] See e.g. my website, PfS, Lecture 1

Archimedes' work shows strikingly how the 'pure' and 'applied', the theoretical and practical aspects of mathematics, need not be at variance but rather can reinforce and enrich each other. It is no accident that of the three greatest mathematicians of all time, Archimedes, Newton and Gauss, all three made distinguished contributions to science as well as to mathematics: the greatest mathematicians tend to be polymaths.

Archimedean Solids (Coxeter Ch. II)

The five Platonic solids have regular polygonal faces *all of the same kind*. If several kinds of face are allowed, thirteen more 'semi-regular' solids are possible, and two infinite families, the 'prisms' and 'antiprisms' (B, 8.12). We know from Pappus (B 11.7) that Archimedes found the complete list (according to Heron, Archimedes ascribed one, the cuboctahedron, to Plato).

1. Truncated tetrahedron, 3.6^2 (one triangle, two hexagons at each vertex)
2. Cuboctahedron, $(3.4)^2$ or 3.4.3.4 (triangle, square, triangle, square)
3. Truncated cube, 3.8^2
4. Truncated octahedron, 4.6^2
5. Small rhombicuboctahedron, 3.4^3
6. Great rhombicuboctahedron = truncated cuboctahedron, 4.6.8
7. Snub cube, $3^4.4$ (laevo and dextro forms)
8. Icosidodecahedron, $(3.5)^2$ (Coxeter, p.19)
9. Truncated dodecahedron, 3.10^2
10. Truncated icosahedron, 5.6^2 ('socer ball')
11. Small rhombicosidodecahedron, 3.4.5.4
12. Great rhombicosidodedcahedron = truncated icosidodecahedron, 4.6.10
13. Snub dodecahedron, $3^4.5$ (dextro and laevo)

See Heath I, Ch. IX, Plato, 294-5, II, Ch. XIII, Archimedes, 98-101; diagrams, Cundy & Rollett 3.7, 100-115. The complete list need not be learned, but you should know that it exists and contains 13 members, including truncated forms of the five Platonic solids.

Carbon 60 and other fullerenes

H. W. (Sir Harry) Kroto (1939-2016) and others discovered a new form of carbon, C_{60}, in 1985 (Kroto, R. Curl and R. Smalley were awarded the 1996 Nobel Prize in Chemistry for this). The C_{60} molecule has carbon atoms at the 60 vertices of a truncated icosahedron. The shape is familiar to those who watch football: typically, soccer balls have 20 white hexagonal and 12 black pentagonal faces. This form of carbon (in addition to diamond and graphite) is called *fullerene*, from the 'geodesic dome' of the architect Buckminster Fuller. It is found in outer space (it has even been suggested that life on earth originated from this source). It has remarkable properties; other forms are C_{70}, C_{76}, C_{82}, C_{84}.

Apollonius
Sources:
H. S. M. COXETER, The problem of Apollonius, *Amer. Math. Monthly* **75** (1968), 5-15;
G. J. TOOMER, Lost Greek mathemtical work in Arabic translation. *Mathematical Intelligencer* **6** (1984), 32-28 (MR 85f:01008).

Apollonius (of Perga, c. 262 – c. 190 BC) completes, with Euclid and Archimedes, the trio of great mathematical figures of the 3rd C. BC, the 'Golden Age of Greek Mathematics'. He was born in Perga in Asia Minor and educated at Alexandria, moving on to Pergamum (Asia Minor), the second centre of learning (after Alexandria) at that time.

While Euclid's gifts lay in elementary *exposition*, (the *Elements* was the first 'scientific best-seller'), Apollonius, like Archimedes, excelled at advanced work. To antiquity, Apollonius was the Great Geometer, Euclid the Elementator.

The *Conics*

Apollonius' principal mathematical work is his *Conics*. Of the eight books, Books I-IV survive in Greek; Books V-VII survived through Arabic and Latin translations; I-VII have been published in modern languages.

We owe to Apollonius our modern picture of conics: sections of any (doubly infinite) cone by a plane, completing the work of Menaechmus (Week 2, B 6.10,11) of the previous century. He also introduced the modern terms
ellipse ('deficiency'; ellipsis: the figure of speech in which words are omitted);
parabola ('comparison');
hyperbola ('excess'; hyperbolae: the figure of speech of exaggeration)
– two-branched, as the cone is doubly infinite.

Apart from the methods of coordinate geometry and projective geometry, which belong to the modern world, Apollonius' treatment of the conics seems complete on all major aspects except for the concepts of focus, directrix and eccentricity. These were introduced by Pappus of Alexandria (B 11.7). Some commentators, such as Zeuthen, believe that the focus-directrix property can be traced to Euclid, in which case its absence in Apollonius is surprising; see Coolidge, Conics, 10, 13, B 9.15.

Book I: Construction and fundamental properties of the three types of conic;
Book II: Diameters, axes, asymptotes, tangents;

Book III: The 3-line problem: let P be a point, L_i be lines ($i = 1, 2, 3$), d_i the distance from P to L_i. Find the locus of P if $d_1^2 = c.d_2 d_3$. The 4-line problem: as above, with $d_1 d_2 = c.d_3 d_4$.
Apollonius was particularly proud of this book, as he stated in the Preface to the *Conics*. He showed by synthetic (classical geometric) methods that both loci are conics. Pappus generalised the problem to more than 4 lines; Descartes applied coordinate methods to this in 1637 (B 9.11).
Book IV: Intersection of conics (B 9.12);
Book V: Extremum properties of tangents and normals (B 9.13);
Book VI: Similarity;
Book VII: Conjugate diameters;
Book VIII: lost ('Appendix': B 9.14).

Perhaps the two great 'might-have-beens' of Greek mathematics are how close they came to discovering calculus and coordinate geometry. We postpone fuller discussion till we take up these topics later.

Other Works

Tangencies (lost). The 'problem of Apollonius' is to construct a circle tangent to three objects, each a point, line or circle. See Coxeter (1968) (above), B 9.3.

Epicycles

Apollonius proposed a model for planetary motion based on epicycles (and one on eccentric motion). The system of Eudoxus (Week 2) and Apollonius dominated thinking on planets and their orbits till the great age of Tycho Brahe, Kepler and Newton. See Neugebauer (1959), Dreyer (1953), B 9.4. The Royal Observatory at Greenwich contains an interesting collection of mechanical devices for predicting planetary orbits based on these schemes.
Comparison of the Dodecahedron and Icosahedron (lost): see Euclid, XIV.
Neusis ('Vergings'), *Cutting-off of an Area*, *Cutting-off of a Ratio*, *Determining a Section*, *Plane Loci* (all lost).

Postscript

The early modern historians of mathematics, such as Heath, studied the Greeks in Greek. However, because much of the Greek heritage only survived through Arabic translation, one should add to this the scholarship of modern mathematicians able to work from the Arabic. See e.g.
G. J. TOOMER, ed. (Springer, 1990): *Apollonius of Perga: Conics, Books V-VII. The Arabic translation of the lost Greek originals in the version of Banu Musa.*
Volume I: Introduction and text. Volume II: Commentary.

A reviewer (J. L. Berggren, Math. Reviews 92e:01017) comments:
"Although Toomer gives full credit to Heath for much that is valuable in his work he quite correctly points out several reasons why Heath's translation is useless for modern historical studies. The work under review is in fact the first appearance of a properly edited Arabic text of the work and a translation based on such a critical edition."

Aristarchus (of Samos, fl. c. 280 BC)
T. L. HEATH: *Aristarchus of Samos: The ancient Copernicus* (1913; reprinted 1981, Dover). By observations made of eclipses, etc., Aristarchus was able to estimate the relative sizes of the earth, sun and moon. His estimates, though highly inaccurate by modern standards, were a good deal better than previous ones. He published his results in a book, 'On the dimensions and distances of the sun and moon'.

Both Archimedes (Dreyer, 138-8) and Plutarch (Dreyer, 138-140) gave detailed accounts of Aristarchus' views on the universe (the modern phrase 'solar system' is perhaps too coloured by hindsight here), in which they assert that Aristarchus pictured the earth as rotating about the sun – the *heliocentric* system of today. As for his book, Dreyer (p. 136) says flatly 'This treatise does not contain the slightest allusion to any hypothesis on the planetary system ...'; Boyer asserts (p. 180) that the book takes a geocentric view. But Heath, in his book on Aristarchus (p. iv) states that:
'... there is still no reason to doubt the unanimous verdict of antiquity that Aristarchus was the real originator of the Copernican hypothesis'.

Thus Aristarchus is a figure of tremendous importance; he has claims to be regarded as 'the father of astronomy', and/or 'the ancient Copernicus'.

Eratosthenes (of Cyrene, 276 – 194 BC) (B 10.3, Dreyer Ch. VIII)

Eratosthenes came to Alexandria in 235 BC as Librarian of the Museum. His most celebrated work is his measurement of the perimeter of the Earth based on comparison of shadows at midsummer noon down wells at Alexandria and Syrene. His figure of 250,000 stades (c. 24,662 miles) compares very well with the modern figures (24, 902 miles at the the Equator, 24, 860 round a great circle through the poles – e.g., the later Greenwich Meridian). This famous experiment is discussed in detail in Dreyer, 174-6.

Eratosthenes is also remembered in Number Theory for the *sieve of Eratosthenes*: find the primes by casting (or sieving) out multiples of 2,3,5,... Recall also that it was to Eratosthenes at Alexandria that Archimedes of

Syracuse sent his *Method*. Eratosthenes also worked on Geography.

Hipparchus (of Nicea, c. 180 – c. 125 BC) (B 10.4)
Trigonometry (trigon = triangle + metria = measurement, Greek)

Although Greeks had long studied arcs and chords, and applied them to, e.g., astronomy, Hipparchus was, it seems, the first to tabulate arc v. chord, thus becoming the 'father of trigonometry.

The general use of the 360^o circle probably stems from Hipparchus' tables (though its origins are Mesopotamian).

Astronomy

Astronomy flourished in Mesopotamia, whence it made its way into the Hellenistic world (e.g. through Berossos, who moved to Cos c. 270 BC). Hipparchus stands between the Babylonian roots and the later achievements of Ptolemy. He discovered the precession of the equinoxes, drew up star catalogues, improved measurements of astronomical constants, and systematised the astronomical heritage of the time (cf. Euclid with geometry).

Menelaus (of Alexandria, c. 100 AD) (B 10.5)

Menelaus' works include *Chords in a circle*, Books I - VI and *Elements of geometry*. His *On spheres* survives, through the Arabic.

'Menelaus' Theorem' (so called, though known already) for plane triangles states that, if a line L cuts the sides BC, CA, AB of a triangle in points D, E, F, then

$$AF.BD.CE = AE.BF.CD.$$

Menelaus, in *Spheres*, extended this result to spherical triangles.

Spherical Trigonometry

One might thus call Menelaus the 'father of spherical trigonometry'. This subject no longer appears in the mathematics curriculum at school or university level. But it is of great practical value in astronomy and navigation (naval and aerial).[3] It is of theoretical value as a first hint of non-Euclidean geometry.[4]

[3]Those who have taken intercontinental flights recently may recall seeing the route displayed on screens. Display of a route on the sphere on a plane screen is a mathematical problem. Route selection is another problem, involving prevailing winds, the need to avoid air traffic using other airports, etc.

[4]Euclidean space is flat – has zero curvature. The sphere has constant positive curvature. The model of non-Euclidean geometry developed in the 19th C. has constant negative curvature.

Ptolemy (of Alexandria, fl. c. 127 – 150 AD); B 10.6-11, Dreyer Ch. IX, Heath II, Ch. XVII.

Almagest

The *Megiste Syntaxis* ('greater collection') of Ptolemy (known to the Arabs as al-majisti, hence Almagest), the classic on trigonometry in the ancient world, had an enormous influence on mathematics and astronomy, lasting till the time of Copernicus 14 centuries later.

Ptolemy's Theorem

If $ABCD$ is a cyclic quadrilateral,

$$AB.CD + BC.AD = AC.BD.$$

The Greeks did not have our trigonometric functions, but expressed in terms of these the theorem leads to the familiar addition theorems for cosine and sine:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

– known consequently as *Ptolemy's formulae.* They imply of course the familiar double angle formulae, etc. Hence Ptolemy was able to extend the work of Hipparchus and construct trigonometric tables (*Almagest*, Book I). As a by-product, Ptolemy obtained an impressively accurate evaluation of π:

$$\pi \sim 3 + \frac{8}{60} + \frac{30}{60^2} = 3.1466666...$$

(this value may have been obtained earlier by Apollonius; see Heath I, 232-5).

Ptolomaic System

The remainder of the Almagest is most noted for the Ptolomaic system of astronomy. Ptolomy took a geocentric view, which seems a retrograde step conceptually. But it has practical advantages – it is still used in planetaria – and may be thought of as representing apparent (or relative), rather than actual, motion.

We refer to Dreyer IX for a detailed account of this system. Innovations included eccentricity, deferent, equant etc.

Geography, Books I-VIII.

Ptolemy's *Geography* was the standard work of its time; Ptolemy may be regarded as the 'father of geography' Included are: latitude and longitude; an estimate of the size of the earth; two systems for map projection,

including stereographic projection.[5] Ptolemy also knew that this was conformal (angle-preserving). The earliest extant mediaeval maps, over a thousand years later, are still based on Ptolemy's work.
Note. Ptolemy badly underestimated the size of the earth, and this had important consequences for Columbus. Thanks to the influence of Henry the Navigator (King of Portugal), the Portuguese knew of Ptolemy's error, and calculated that the ships of the time could not carry adequate stores to reach India by sailing West. They concealed this knowledge from the Spaniards, who dispatched Columbus in 1492.[6] On discovering the New World, he thought he had found India, whence our term West Indies[7]; pre-Columbian Americans (North and South) are still loosely called 'Indians'.
Optics (B. 10.11): theory of mirrors; refraction.
Tetrabiblos (= *Quadripartitum*). This is a work of astrology, which it is amazing, by modern standards, to find written by the same author as the *Almagest*, a superb work of science. One must remember the Babylonian roots of both astronomy and astrology, and the position of Alexandria in the Near East; one may perhaps also look ahead to the decline and eclipse of Greek mathematics and science.

Heron (of Alexandria, 3rd C. AD) (B 10.12-14; Heath II, Ch. XVIII)

There is uncertainty as to Heron's dates (see Heath 298-307 for the evidence) – probably the 3rd C. AD. Also used is Hero, the Roman form of Heron.
Heron's formula: if a triangle has sides a, b, c, semi-perimeter $s := \frac{1}{2}(a+b+c)$ and area Δ, then

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

For a modern (trigonometric) proof: $\Delta = \frac{1}{2}bc\sin A$. Use $\sin^2 A = 1-\cos^2 A = (1-\cos A)(1+\cos A)$ and the cosine rule $a^2 = b^2+c^2-2bc\cos A$ to substitute for $\cos A$ and simplify. The result is known, from Arab commentators, to be due to Archimedes (Sigler's Law again), but the terminology is traditional, established and useful.
Metrica
Book I: Measurement of areas (including Heron's formula with proof). Quadri-

[5] See e.g. my homepage, link to M2P3 Complex Analysis, Lecture 4.
[6] Some of Columbus' sailors feared that they might reach the edge of the Earth, and fall off.
[7] and Windies for the West Indian test cricket team

laterals; circles; regular polygons; approximate evaluation of square roots.
Book II: Measurement of volumes. Cones, cylinder, prism, pyramid, frustrum, torus, Platonic solids.
NB. The volumes of the Platonic solids are given exactly in Euclid, Apocrypha. Heron gives only approximate answers, but – here and elsewhere – does not distinguish between the exact and the approximate. This defect is more typical of pre-Greek than of Greek mathematics proper, and is a clear sign of a lowering of standards (B Ch. 11).

Other works on mensuration include the *Geometrica*, *Stereometrica*, *Deodesia*, *Mensurae*. He also wrote on optics, mechanics and the construction of engines of war.
Catoptrica.

Heron showed by a simple geometric argument that the law of reflection (angle of incidence = angle of reflection) is equivalent to assuming that light follows paths of *shortest distance*. This is *Heron's Principle*.

In a homogeneous medium, this says that light follows paths of *shortest time*. This result, known as *Fermat's Principle* (B Ch. 17) is the cornerstone of Geometric Optics. Of course, light was thought to have infinite speed in the ancient world (indeed, until the 17th C.).

Diophantus (of Alexandria, fl. c. 250 AD) (B Ch. 11; Heath, *Diophantus*; Heath II Ch. XX).

Diophantus is regarded as the 'father of algebra'; he went for beyond his Mesopotamian predecessors in algebra. He was only interested in exact solutions of equations, and worked rigorously with proofs. He restricted himself to positive rational solutions of equations; here, he solved quadratic equations completely, and some cubics, simultaneous quadratics, etc.

Diophantus' name is best remembered today in the term *Diophantine*, in which one has *indeterminate* equations (one with more unknowns than equations – so infinitely many solutions) and seeks *integer* solutions. This subject (discussed at length in Heath II, 466-476, 490-514) is now regarded as part of Number Theory rather than Algebra. It was taken up in the 17th C. by Fermat (B Ch. 17), and is now a thriving modern field.[8] See e.g.
L. J. MORDELL, *Diophantine equations*, Academic Press, 1969.

[8]Fermat's Last Theorem, famously proved by (Sir) Andrew Wiles (1953-; K, 2000) in 1995.

Pappus (of Alexandria, fl. c. 290 AD) (B 11.7-10, Heath II, Ch. XIX).

So far as his original mathematics is concerned, Pappus is remembered for: *Focus-directrix(-eccentricity)* theorem for conics (Coolidge, *Conics and quadrics*, 8-13): If F (focus) is a point, L (directrix) is a line, e (eccentricity) is a positive constant, the locus of a point P such that its perpendicular distance PL to the line L satisfies

$$PF = e.PL$$

is a conic section. (With conics defined as previously as conic sections, this is a theorem. But it can be used as a definition of conic, when the previous definition becomes a theorem.)[9] This contribution completes the Greek work on conics of Apollonius and others, and is the last major result before modern times and methods (analytic and projective).

Pappus' Theorem. Let A, B, C be colinear points on a line L, A', B', C' be colinear on L', and let L, L' meet in O. If BC', $B'C$ meet at D, CA', $C'A$ at E and AB', $B'A$ at F, then D, E, F are colinear.

The Collection or *Synagoge*. Heath (357-8) writes 'Obviously written with the object of reviving the classical Greek geometry, it covers practically the whole field. It is, however, a handbook or guide to Greek geometry rather than an encyclopaedia; it was intended, that is, to be read with the original works (where still extant) rather than enable them to be dispensed with.'

Books I, II: lost.

Book III: proportions; theory of means; inscribing Platonic solids to a sphere.

Book IV: Extension of Pythagoras' theorem (B 11.9); use of curves (spirals, quadratrix etc.) in circle-squaring and angle-trisection.

Book V: Isoperimetry. By this is meant the subject typified by the (much later) theorem that of all curves of a given length, that enclosing the greatest area is a circle. Pappus shows that for regular n-gons of given perimeter, the area increases with n. He applies this in his famous comment on the 'sagacity of bees' (beginning of B Ch. 11) for making hexagonal rather than square or triangular honeycombs. [10] Book VI: Astronomy; optics.

Book VII: Description and commentary on works of Euclid, Apollonius etc. Focus-directrix property of conics. Volumes of solids of revolution (B, 11.12).

---

[9]The focus-directrix property of conics is the one we use to show Newton's epoch-making result from the *Principia*: the Inverse Square Law of Gravity is *equivalent* to orbits being conic sections (elliptical in our case: the Earth's orbit round the Sun is closed.

[10]Compare the occurrence in geology of hexagonal columns of basalt, as in the Giants' Causeway of N. Ireland, formed when molten rock cools.