m3hw9.tex **Week 9. 8-13.3.18**

From the 19th C. to the 20th C.

Analysis

While great strides have been taken in the Analysis so far, modern standard of rigour had not yet been achieved. These emerged gradually during the 19th C.

Bernhard Bolzano (1781-1848), theologian of Prague
Rein analytischer Beweis ... (1817) (Purely analytical proof).

This gave a proof of the Fundamental Theorem of Algebra, improving on that of Gauss (recall that this result is one of Analysis!). It gave the modern definition of continuity. It gave the 'Cauchy condition' for convergence of a sequence of reals – before Cauchy (Week 8), It showed that a set of reals bounded above has a supremum (least upper bound).
Bolzano's Theorem: If f is continuous on $[a,b]$ and $f(a)$, $f(b)$ have opposite-signs, then $f(c) = 0$ for some $c \in (a,b)$.
Paradoxien des Unendlichen (1851, posth.). Here Bolzano, in drawing attention to the need to *compare* infinite sets according to their 'size', helps to prepare the ground for Cantor's work (below).

We have already discussed the work of Dirichlet and Riemann (Week 8).

Karl Weierstrass (1815-1897), Professor of Mathematics at Berlin (1864).

Weierstrass, a pupil of Christof Gudermann (1798-1852), was a provincial schoolmaster (in Braunsberg), and became famous overnight for an important paper on Abelian functions published (in *Crelle's J.*, or *J. reine ang. Math.*) in 1854. He became an Assistant Professor in Berlin, and taught there for the rest of his career.

He was an inspired and influential teacher. The undergraduate curriculum in Analysis has a more obvious debt to Weierstrass than to anyone else – in particular, the introduction of the epsilon-delta technique, and modern standards of rigour, is basically due to him. So too is the idea of *uniformity* (uniform convergence, uniform continuity, etc.).

Much of Weierstrass' work remained unpublished till his Collected Works (*Werke*) apeared, and much was published under the names of his pupils. Thus his work is difficult to date; his greatest impact lay through his lectures, his teaching, and his influence on others.

Real Analysis
Bolzano-Weierstrass Theorem: Weierstrass, in his 1860s lectures, completed Bolzano's work.
Functions continuous on closed intervals. Weierstrass' lectures also proved the first three of the four fundamental properties of functions f continuous on closed intervals $[a, b]$:
1. f is bounded;
2. f attains its bounds;
3 (Intermediate Value Theorem). f attains every value between its bounds. [Apply Bolzano's Theorem to $f - c$, where $\inf\ f < c < \sup\ f$.]
Weierstrass Approximation Theorem (1885): If f is continuous on $[a, b]$, f may be approximated (arbitrarily closely) by polynomials, *uniformly* on $[a, b]$.
Comparison Test, or Weierstrass M-Test, for convergence of series.
Continuous nowhere differentiable functions. Weierstrass (1860s) gave an example of an everywhere continuous, nowhere differentiable function. Such an example had been given earlier by Bolzano; for other examples, see e.g. Kline, 955-6. These examples showed convincingly that continuity does *not* imply differentiability, *anywhere*, and hence that arguing from graphs and diagrams using naive geometric reasoning is quite unreliable in Analysis. It is from this time on that rigorous analysis ('university-style', without diagrams, in contrast to 'school-style') really took off.
Complex Analysis

Weierstrass took as his motto 'Everything is power series'. The Cauchy-Taylor theorem of Complex Analysis tells us that analyticity (differentiability in the sense of Complex Analysis) is the same as holomorphy (being expressible by a power-series expansion at every point) (so the two terms can be used interchangeably). Weierstrass built on this, identifying an *analytic function* $f(z)$ with the totality of its power-series expansions $\sum_0^\infty a_n(z - z_0)^n$ representing f around each point z_0 at which it is analytic. Hence the concept of *analytic continuation.* This is emphasised in, e.g., NHB, M2P3 and M3P16. Hardy (*Divergent Series*, Ch. I) points out that what Euler really lacked was the concept of analytic continuation.

Eduard Heine (1821-1881).
Heine's Theorem (1871): If f is continuous on $[a, b]$, f is *uniformly* continuous on $[a, b]$ ('4', completing 1-3 of Weierstrass).
Open coverings. Following work in 1895 by Emile Borel (1871-1956), Heine's theorem and its proof led to the modern

Heine-Borel Theorem: Any (infinite) class of open sets covering (i.e., whose union includes) $[a, b]$ contains a *finite* class which covers $[a, b]$ (a finite subcover). This 'Heine-Borel property' is the germ of the concept of *compactness*, one of the key concepts in *topology* (20th C.).

REAL NUMBERS

One of the surprising facts about the history of mathematics is that, although mathematicians had been aware of the irrationals since the Greeks, no satisfactory construction of the real number system – of the irrationals from the rationals – had ever been given. The year 1872 saw *two distinct* such constructions, due to Dedekind and Cantor. Dedekind's is perhaps the simpler, but it is specific to the real number system (real line) $\mathbb{R}$, because it involves the idea of *order*, technically the *total ordering* of the reals: if $a, b \in \mathbb{R}$, exactly one of $a < b$, $a = b$ $a > b$ holds. The other, due to Cantor, is generally applicable, and its generalisation comes in the 20th C.

Richard Dedekind (1831-1916), Professor, Technische Hochschule, Braunschweig, 1862.
Stetigkeit und die Irrationalzahlen (1872) [Continuity and the irrational numbers]

Dedekind's construction of $\mathbb{R}$ centres on the idea of a *Dedekind cut* or *section* (Schnitt). The irrational $\sqrt{2}$ divides the rationals $\mathbb{Q}$ into two classes, those $< \sqrt{2}$ and those $> \sqrt{2}$, either of which serves to identify $\sqrt{2}$. For a brief treatment, see e.g.
G. H. HARDY, *A course of pure mathematics*, 10th ed., 1952, CUP, Sec. 17.
Infinite sets

We saw that Galileo observed that an infinite set can be put into one-one correspondence with itself. Dedekind used this property to *define* an infinite set. Thus a set is (Galileo-Dedekind) infinite iff it can be put into one-one correspondence with a proper subset of itself.
Dirichlet's Pigeonhole Principle (1837) states that, for A, B finite sets of the same cardinality, $f : A \to B$ is injective iff it is surjective. That is, a set is infinite iff it violates the Dirichlet Pigeonhole Principle. This Principle is important in Combinatorics; see e.g.
P. J. CAMERON, *Combinatorics: Topics, techniques, algorithms*, CUP, 1994, Ch. 10, 19.
Was sind und was sollen die Zahlen (MS 1872-8; publ. 1888) [What numbers are and what they should be.] A foundational study of $\mathbb{Z}$.

Georg Cantor (1845-1918)
Cantor's construction of the reals (*Math. Annalen* **5** (1972), 123-132 and **21** (1883), 545-591).

Call a sequence $a = (a_n)$ of rationals *fundamental*, or *Cauchy*, if $a_{m+n} - a_n \to 0$ $(m, n \to \infty)$, i.e.

$$\forall \epsilon > 0 \quad \exists N = N(\epsilon) \ s.t. \ n > N, \ m \geq 0 \ \Rightarrow \ |a_{m+n} - a_n| < \epsilon.$$

Call $a = (a_n)$, $b = (b_n)$ *equivalent* if $a - b := (a_n - b_n) \to 0$ (this *is* an equivalence relation – check). Cantor *defines* a real number to be an *equivalence class of Cauchy sequences of rationals.* For details, see e.g. (for $\mathbb{R}$)
B. L. van der WAERDEN, *Modern Algebra*, Vol. 1, Ch. 9,
and for the more general setting of a metric space (20th C., below),
J. C. & H. BURKILL, *A second course in math. analysis*, CUP, 1970, 3.5.

Completeness

We call a metric space (below) *complete* if every Cauchy sequence is convergent. Thus $\mathbb{Q}$ is not complete (the decimal approximants to $\sqrt{2}$ do not converge in $\mathbb{Q}$), but by Cantor's result $\mathbb{R}$ is complete, and so is $\mathbb{C}$, etc. Not only can $\mathbb{Q}$ be *completed* to obtain $\mathbb{R}$, but *every* metric space can be so completed. This generality of Cantor's construction of $\mathbb{R}$ by completion and Cauchy sequences makes it preferable to Dedekind's via cuts.

Countable and uncountable sets

Cantor's greatest achievement was the realisation that infinite sets can be usefully classified according to their 'size'. First, the simplest case.

Definition (Cantor). An infinite set S is *countable* (or *denumerable*) if it may be put in one-one correspondence (bijection) with $\mathbb{N}$, *uncountable* otherwise.

Thus the elements of a countable set S can be *listed*, $\{x_1, x_2, \ldots, x_n, \ldots\}$ ('listable' would perhaps be preferable to 'countable'). So $\mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$ is countable. One can easily show ('diagonal sweep': Cantor, 1885) that the union of countably many countable sets is countable (so $\mathbb{Q}$ is countable), and hence (Cantor, 1895) that $\mathbb{R}$ is uncountable. As a corollary, *irrational numbers exist* (or $\mathbb{R}$ would be countable). Furthermore, in a sense that these results make precise, 'most' reals are irrational: *rationals are flukes.* But we can see this from decimal expansions. A real is rational iff its decimal expansion terminates or recurs. Why should it? This is exceptional behaviour.
[By contrast: for continued fraction expansions: finite iff rational, recurring iff a *quadratic irrational* (Lagrange's theorem: H&W 10.12).]

ALGEBRAIC AND TRANSCENDENTAL NUMBERS

The irrational $\sqrt{2}$ is a root of the polynomial $x^2 - 2$, and similarly for other 'surd-like' irrationals. Call a real number *algebraic* if it is a root of a polynomial with integer coefficients, *transcendental* otherwise.

Theorem (Cantor, 1874). The algebraics are countable (and so the transcendentals are uncountable).

Proof. The algebraics are $\cup_{n=1}^{\infty} A_n$, where A_n is the set of roots of polynomials of degree n with integer coefficients. There are only countably many such polynomials $p(x) = a_0 x^n + \ldots + a_n$; each has (at most) n real roots (Fundamental Theorem of Algebra), so each A_n is countable, so their union is countable.

Corollary. Transcendental numbers exist! [Most reals are transcendental.]
This is a very fine example of a *non-constructive existence proof* – valuable, because it is *easy*.

The 'obvious' – or constructive – way to show that transcendentals exist is to exhibit an example, but this is *much harder*. Joseph Liouville (1809-1882) showed in 1844 that decimals of the form $\sum a_n/10^{n!}$ are transcendental. Charles Hermite (1822-1901) showed in 1873 that e is transcendental. Ferdinand Lindemann (1852-1939) showed in 1882 that π is transcendental. As a corollary, *it is impossible to square the circle* (constructible numbers are algebraic; if the circle could be squared, $\sqrt{\pi}$ would be algebraic, so π would be algebraic).

Note. 1. This finally settles the ancient Greek problem (Anaxagoras).
2. This convincingly shows the power of Cantor's new set theory.
3. The same argument shows that the set of *computable numbers* is countable – so, most numbers are non-computable. For (though this is a 20th C. concept, due to Turing), a number is *computable* if it could in principle be the print-out from running a programme. But a programme is a finite string of symbols drawn from a finite alphabet; there can be at most countably many such (not finitely many – no upper bound on programme length), so there are only countably many computable numbers. Of course, π is computable (from e.g. Brouncker's continued fraction), even though transcendental.

SET THEORY, LOGIC and FOUNDATIONS

George Boole (1815-1864), Prof. Math., Queen's College, Cork, 1849.
The mathematical analysis of logic (1847);
Investigation of the laws of thought (1854).

Boole's work marks the beginning of mathematica logic proper. It has led to the subject of *Boolean algebra*, in algebra and computer programming.

Augustus De Morgan (1806-1871), first Professor of Mathematics, UCL, 1828; first President LMS (1865-9).

De Morgan is remembered for *De Morgan's laws* in Set Theory. The LMS Headquarters in Russell Square is named De Morgan House.

John Venn (1834-1923)

The logic of chance (1866); *Symbolic logic* (1881).

Venn taught Moral Science at Cambridge from 1862. He is remembered for *Venn diagrams* in Set Theory.

Cantor wrote *Grundlagen einer allgemeine Mannigfältigkeitslehre* (1883) [Foundations of a general theory of manifolds]. Here, and in a series of papers in *Mathematische Annalen* between 1879 and 1884, Cantor set up a general theory of sets, and in particular of infinite ('transfinite') numbers, of two kinds, *cardinal*, to do with *size* (one, two, ...) and *ordinal*, to do with *order* (first, second, ...). We shall return to set theory in the 20th C. We note here that Cantor's theory was violently attacked by Leopold Kronecker (1823-91), whose dictum was 'God gave us the integers; all the rest is the work of man'. But Cantor's views eventually prevailed (rightly); in particular, he was memorably defended by David Hilbert (1862-1943), who said 'No one shall expel us from the paradise that Cantor has created for us'.

Following Cantor's set theory, and the construction of $\mathbb{R}$, the question arose of setting up the *foundations of mathematics* – in particular, $\mathbb{Z}$ – 'from scratch', and by the new standards of rigour. Dedekind gave one approach (above). Another, more usual nowadays, was given by Giuseppe Peano (1858-1932) in 1889. Incidentally, Peano introduced the standard set-theoretic notation we use today. An excellent treatment of 'what the mathematician in the street should know about set theory and foundations' is in

P. R. HALMOS, *Naive set theory*, Van Nostrand, 1960.

ALGEBRA

We have seen determinants (Gauss), and the emergence of groups (Abel) and fields (Galois). Rings and ideals grew out of Algebraic Number Theory (E. E. Kummer (1810-1893), Dedekind and Kronecker). Linear Algebra (vector spaces, linear transformations, linear independence etc.) grew out of Geometry; an important step was by Hermann Grassmann (1809-77), *Ausdehnungslehre* (1844) [Theory of extensions].

Matrices logically precede determinants, but came much later, in the work of Arthur Cayley (1821-95) and J. J. Sylvester (1814-97). The term matrix was introduced by Sylvester in 1850, but a series of papers by Cayley in the 1850s, particularly in Phil. Trans. 1858, created the basis of the subject as we now know it.

Hamilton introduced *quaternions* in 1843 ($a+ib+jc+kd$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, etc.) – a non-commutative division ring. The *Cayley-Hamilton theorem* (a matrix satisfies its own characteristic equation) is from Hamilton's *Lectures on quaternions* (1853) and Cayley's 1858 paper.

Gibbs pioneered the use of *vectors*. Though quaternions were used 'for everything' by some, by the 20th C. they had been superceded by vectors.

Group representations enable groups to be studied as groups of matrices, so enabling Linear Algebra to be brought to bear. The theory was developed by F. Georg Roebenius (1849-1917) from 1897, William Burnside (1852-1927) (*Theory of groups of finite order*, 1911).

APPLIED MATHEMATICS and PHYSICS

George Green (1793-1841)

An essay on the application of mathematical analysis to the theories of electricity and magnetism (1828; facsimile 1958).

Mathematical papers of George Green, Chelsea, 1871/1970.

Green was self-taught, and is one of the most remarkable self-taught mathematical geniuses in history. His father owned a flour mill in Sneinton, Notingham. Green worked in the mill, and learned mathematics in a public library in Nottingham. He achieved recognition after his *Essay* of 1828, and became a Fellow of Gonville and Caius College, Cambridge. His health failed, [1] and he returned to Nottingham, where he died. Green's Mill is now a museum. Newton's memorial stone in Westminster Abbey is now flanked by five smaller ones (Green, Faraday, Maxwell, Kelvin, Dirac).

Green's *Essay* introduced the term *potential*. He also proved Green's theorem, and the divergence theorem ('Gauss' theorem), in Vector Calculus (see e.g. NHB, MPC2, VII). *Green functions* are of great importance in many areas (solving PDEs by integrals) and physics (classical and quantum).

Sir William Rowan Hamilton (1805-1865), Professor of Astronomy, Trinity College Dublin, 1827-65.

J. L. SYNGE, *Geometrical optics: An introduction to Hamilton's method*, Cambridge Tracts 37, CUP, 1937.

Hamilton's early work is on *geometrical optics*. This concerns light *rays*

[1] probably a combination of taking in too much flour dust in the mill and too much wine in College

(paths followed by particles or thin beams of light). These are everywhere *orthogonal* to the *wavefronts* of Huygens' theory. Hamilton's optical theory was presented to the Royal Irish Academy in 1827, and resulted in his election to the TCD Chair of Astronomy as an undergraduate of 22. This carried with it the title of Astronomer Royal of Ireland and the directorship of the Dunsink Observatory near Dublin, where Hamilton spent his working life.

Hamilton later worked on dynamics, discovering in particular a close analogy between mechanics and optics.
Hamilton's Principle (1834): With L the Lagrangian, $\int Ldt$ is an extremum. This includes the *Principle of Least Action*, which can be traced back to Maupertuis (1744) (Whittaker, AD, Ch. IX, Sec. 99).
Hamilton's Equations (1834). With $L = L(q_r; \dot{q}_r; dt)$, $p_r := \partial L/\partial \dot{q}_r$ (so $\dot{p}_r = \partial L/\partial q_r$ by Lagrange's equations. The *Hamiltonian*

$$H(p_r; q_r; t) := \sum \dot{q}_r p_r - L$$

satisfies *Hamilton's equations*

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r}.$$

One passes from Lagrangian to Hamiltonian mechanics by the Legendre transform (Goldstein, 7.1). Hamilton's work was continued by Jacobi in 1836, resulting in the *Hamilton-Jacobi theory*:
Jacobi, *Vorlesungen über Dynamik*, 1866.

Hamilton's optical-mechanical analogy prefigures the *wave-particle duality* of *Quantum Mechanics*. See Goldstein, 9.8, especially p.312-4. What was needed was the realisation that 'Planck's constant is positive' (below).
René DUGAS, *A history of mechanics*, Dover, 1988, 662p;
D. D. HOLM, *Geometric mechanics, I, II*, Imperial C. Press, 2008.

Rudolf Clausius (1822-1888) *and Thermodynamics*
Über die bewegende Kraft der Wärme (1850) [On the moving force of heat]

Clausius closes with the most famous two-sentence passage in the history of science:

Die Energie der Welt ist konstant.

Die Entropie der Welt strebt einem Maximum zu.

[The energy of the world is constant. The entropy of the world strives towards a maximum. (Read 'universe' for 'world' in each here.]

These are the First and Second Laws of Thermodynamics. The first is the Law of Conservation of Energy. The second is that *entropy increases*. Entropy is a measure of disorder: nature moves towards disorder.[2]
Sir George Stokes (1819-1903)
Stokes' theorem in Vector Calculus, 1854 ('theorems of Stokes, Gauss and Green: NHB, MPC2, VII).
Michael Faraday (1791-1867) [Whittaker, Ch. VI, Faraday].
James HAMILTON, *Faraday: The life*, HarperCollins, 2002.

Faraday's great discovery (following earlier work by Sir Humphrey Davy) was *electromagnetic induction*: movement of a magnet near a conductor induces an electric current in the conductor. Hence the electric motor (turning electrical energy into mechanical energy) and the dynamo (doing the reverse).

Faraday lectured at the Royal Institution (founded 1799), where he was Professor of Chemistry. The old £20 note shows Faraday giving the Christmas lectures there, initiated 1826.
James Clerk Maxwell (1831-1879) [Whittaker, Ch. VIII: Maxwell].
A treatise on electricity and magnetism, Vol. 1, 2, OUP, 1891/1998.
C. W. F. EVERITT, *James Clerk Maxwell: Physicist and natural philosopher*, Charles Scribner, 1975.
Maxwell's Equations. If E, H are the electric intensity and the magnetic field in ES (electrostatic) units, cE in EM (electromagnetic) units, Maxwell's equations (in a vacuum) are

$$\text{div } E = 0, \quad \text{curl } E = -c^{-1}\partial H/\partial t; \quad \text{div } H = 0, \quad \text{curl } H = c^{-1}\partial E/\partial t.$$

Whitaker (p. 245) writes: 'In this memoir (of 1865) the physical importance of the operators curl and div first became evident. These operators had, however, occurred frequently in the writings of Stokes ...'. Applying curl (recall curl curl = grad div - ∇^2, $= -\nabla^2$ here):

$$-\nabla^2 E = \text{curl curl} E = -\frac{1}{c}\frac{\partial}{\partial t}(\text{curl} H) = -\frac{1}{c2}\frac{\partial^2 E}{\partial t^2}: \qquad \nabla^2 E = c^{-2}\partial^2 E/\partial t^2,$$

and similarly

$$\nabla^2 E = c^{-2}\partial^2 E/\partial t^2.$$

This is the *wave equation*, for propagation of E, H with velocity c, the ratio of EM to ES units. This was known experimentally (c. 3×10^{10} cm/sec.,

[2]Thermodynamics is at the heart of heat transfer, which is a large part of Chemical Engineering.

c. 186,000 miles/sec.) to be (approx.) the *speed of light*. Thus, *electromagnetic forces are propagated with the speed of light.* This suggested to Maxwell his great discovery: that *light waves are electromagnetic*. Recall the modern *electromagnetic spectrum*: in increasing order of wavelength, ..., x-rays, ultra-violet, visible spectrum, infra-red, radio waves, ...

Note. Faraday's discovery of electromagnetic induction and Maxwell's electromagnetic theory of light are regarded as the two greatest advances of 19th C. Physics.

Maxwell is one of the three founding fathers of Statistical Mechanics (or Statistical Physics) (with Boltzmann and Gibbs, below), obtaining the *Maxwell-Boltzmann law* (Maxwell 1859, Boltzmann 1877) for the velocities of particles in a gas.

Lord Kelvin (Sir William Thomson) (1824-1907; K 1866; Baron Kelvin 1892); PRS 1890; 18th President LMS (1898-1900)

W. Thomson & P. G. Tait, *A Treatise on Natural Philosophy*, 1867 ("T and T"; cf. the *Principia*).

Thomson's best-known achievement (1866), for which he was knighted, was the laying of the first successful Atlantic cable; the relevant mathematics is the *equation of telegraphy*, a PDE containing both the heat equation and Laplace's equation. He also introduced the *method of images* in EM theory (based on projective geometry). He also showed that the equilibrium distribution in electrostatics is that which *minimises the energy* (cf. CoV).

Kelvin in his later years was a conservative figure; in particular, he opposed Maxwell's electromagnetic theory of light (Whittaker, EM, 266-7).

Lord Rayleigh (J. W. Strutt) (1842-1919), 7th President LMS (1876-8), PRS (1905-8), Nobel Prize 1904.

The theory of sound, vol. I, II, 1877 (Dover, 1945)

Rayleigh's work on *scattering* explained why the sky is blue (the earth's atmosphere scatters sunlight – the sky is black in space). His Nobel Prize was for the discovery of the inert gas argon. He also worked on *tides* (which show periodicities, for which Fourier methods are well suited).

Kelvin and Rayleigh also worked on hydrodynamics. For details and references, see e.g.

H. LAMB, *Hydrodynamics*, CUP, 1924.

Josiah Willard Gibbs (1839-1903)

Gibbs was one of the pioneers of American science. In Hamiltonian mechanics, with n particles ($H = H(p_1, \ldots, p_n; q_1, \ldots, q_n; t)$), new methods are needed to extract useful information from Hamilton's equations as $n \to \infty$.

Gibbs introduced the limiting density for such a system, the *Gibbs distribution*: density $f(p,q) = c\exp\{-\beta H\}$, where c is the constant such that f integrates to 1, H is the Hamiltonian, and $\beta = 1/(kT)$ with T the temperature (absolute – in degrees Kelvin) and k is Boltzmann's constant (below).

The underlying principle here is the *Gibbs Variational Principle*. This has profound implications for modern physics and Probability Theory (large deviations); for details, see e.g.

Hans-Otto GEORGII, *Gibbs measures and phase transitions*, W. de Gruyter, 1988;

Richard S. ELLIS, *Entropy, large deviations, and statistical mechanics.* Grundl 271, Springer, 1985.

Ludwig Boltzmann (1844-1906)

D. LINDLEY, *Boltzmann's atom: The great debate that launched a revolution in physics.* Free Press, 2001.

Boyle's Law for a gas is $PV =$ constant. For an ideal gas with n particles, $PV = nkT$, where P is pressure, V is volume, T is temperature and k is *Boltzmann's constant*. Boltzmann also refined Maxwell's work on velocity distribution in gases (Maxwell-Boltzmann law), and Clausius' Second Law of Thermodynamics (Boltzmann's H-theorem – H for entropy).

Boltzmann's work was on gases, and he clearly understood that gases are composed of atoms and molecules. But these could not then be directly observed experimentally, and the atomic theory was opposed by some conservative scientists. Boltzmann became embroiled in controversy; this undermined his mental health, and eventually drove him to suicide.

Henri Poincaré (1854-1912), Professor of Mathematics, Sorbonne, Paris

Nouvelles Méthodes de las Mécanique Céleste, Vols. I-III (1892/93/99);

Lecons de Mécanique Céleste, I-III (1905-10).

Poincaré contributed extensively to the *three-body problem* in the theory of orbits. He had a life-long interest in the *stability of the solar system.* This led him into the *qualitative theory of ODEs* (Poincaré-Bendixson Theorem). *Poincaré's Recurrence Theorem* of 1899 concerns *ergodic theory* (originally, the long-term behaviour of dynamical systems).

Analysis Situs (1895)

This was an early book on the subject we now call *topology.* Here Poincaré introduced *homology*, *homotopy* and the *fundamental group.* See e.g. NHB, M2P3, Handout, Variants on Cauchy's theorem.

Hyperbolic geometry. Poincaré gave his model for non-Euclidean geometry on the disc. For a visual interpretation, see *Circle Limit IV* by the Dutch

artist Maurits Cornelis Escher (1898-1972) [W].
David Hilbert (1862-1943), Professor of Mathematics at Göttingen, 1895.

Hilbert came from Königsberg, where he was a pupil of Heinrich Weber (1842-1913); he succeeded Weber in Göttingen. His doctoral thesis of 1885 is on invariants. He proved the Hilbert Basis Theorem in 1888, and the Hilbert Nullstellensatz.
Zahlbericht (1897) [Report on Numbers]. This book may be taken as the starting-point for the modern subject of Algebraic Number Theory; a key contributors to this was Hilbert's pupil E. Hecke (1887-1947).
Grundlagen der Geometrie (1899) [foundations of Geometry]. Hilbert's book was an attempt to begin geometry axiomatically from scratch, by modern standards of rigour and with modern knowledge (such as non-Euclidean geometry) – "to bring Euclid's Elements up to date". Hilbert took the modern view of mathematics as *deductive reasoning from axioms*, and emphasised the importance of the *axiomatic method*, which we take for granted today. However, Hilbert's views on the foundations of mathematics were later shown to be too naive (see Week 10).
The Hilbert Problems.

Hilbert proposed a famous list of (23) problems to the International Congress of Mathematicians in Paris in 1900. These have been extremely influential. The solution of a Hilbert problem is a major achievement in mathematics; for details of progress, see e.g. *Math. developments arising from Hilbert problems*, Proc. Symp. Pure Math. XXVIII, AMS, 1976.
The beginnings of functional analysis

Hilbert's pupil Erhard Schmidt (1876-1959) took his PhD in 1905. He worked on integral equations in *Hilbert space*, which he named – basically, the extension of Euclidean space to infinitely many dimensions. *Gram-Schmidt orthogonalisation* followed in 1907 and *Hilbert-Schmidt* operators in 1908.

Hilbert's pupil Richard Courant (1888-1972) took his PhD in 1910 (Dirichlet's principle and conformal mapping). From Hilbert's lecture notes, Courant wrote 'Courant and Hilbert':
R. COURANT & D. HILBERT, *Methoden der mathematischen Physik*, I (1st ed. 1924, 2nd ed. 1931), II (1937).
This classic book arrived just in time to serve the needs of the new subject of Quantum Mechanics.