M2PM3 HANDOUT: THEOREMS OF BOLZANO & WEIERSTRASS, CANTOR AND HEINE

The contents of this handout are *not examinable*. The first two results are included (they are closely related) as either can be used as the main step in the proof of the main result of this course, Cauchy's Theorem (II.5). The third (whose corollary we quoted in lectures) is included as a simple illustration of the power of compactness. Recall that by the Heine-Borel Theorem which we quoted, 'compact' is the same as 'closed and bounded' in this course. So we can (and often do) say 'closed and bounded' when what really matters is 'compact'; here we spell out one compactness argument, to illustrate the definition in action.

Theorem (BOLZANO-WEIERSTRASS). If S is a bounded infinite set in the complex plane or Euclidean space, S has a limit point.

Proof. We set out the proof in the plane, and use *repeated bisection.* As S is bounded, we can enclose it in a large square, T say, of side L, with sides parallel to the coordinate axes. Divide T into four equal subsquares by bisecting each side. As S is infinite and $S \subset T$, T contains infinitely many points of S. So at least one of the four subsquares also contains infinitely many points of S; call this T_1. Bisect again and divide T_1 into four equal subsquares. As above, at least one of these contains infinitely many points of S; call this T_2. Continuing in this way, we obtain an infinite sequence $\{T_n\}$ of squares, with $T \supset T_1 \supset \ldots \supset T_n \ldots$ and each T_n of side $L/2^n$ and containing infinitely many points of S.

Let $[a_n, b_n]$ be the projection of T_n on the x-axis. The a_n are non-decreasing, and bounded above (by b_1), so increase to a limit, a say. Similarly the b_n decrease to a limit, which is a as $b_n - a_n = L/2^n \downarrow 0$. Similarly, if $[c_n, d_n]$ is the projection of T_n on the y-axis, c_n increases to a limit c, and d_n decreases to c. Write P for the point (a, c) and choose $\epsilon > 0$. For all n large enough (so that $L/(2^n\sqrt{2}) < \epsilon$), T_n is contained in the disc $N(P, \epsilon)$ centre P radius ϵ. As T_n, and so the disc, contains infinitely many points of S, P is a limit point of S. //

We call a sequence of sets S_n *decreasing* if $S_1 \supset S_2 \ldots \supset S_n \supset S_n \ldots$.

Theorem (CANTOR: Nested Sets Theorem). If K_n is a decreasing sequence of non-empty closed and bounded (compact) sets in the plane (or Euclidean space), their intersection $\cap_n K_n$ is non-empty. If the diameters $diam(K_n) \downarrow 0$, the intersection is a single point.

Proof. As above, let K_n have projections $[a_n, b_n]$, $[c_n, d_n]$ on the coordinate axes (closed and bounded, and also decreasing, as K_n are). Then again as above, $a_n \uparrow a$, $b_n \downarrow b$, $c_n \uparrow c$, $d_n \downarrow d$ for some a, b, c, d. Then (e.g.) the point (a, c) is in $\cap_n K_n$. If the diameters decrease to 0, $a = b$, $c = d$ and this is the *only* point in the intersection. //

Note. 1. Cantor's Theorem is the heart of the proof of the main theorem of the course, Cauchy's Theorem (II.5).
2. Recall that in a metric space compact implies closed and bounded, and by the Heine-Borel Theorem, in Euclidean space the converse holds: 'closed and bounded implies compact'.
3. The proof we have given uses 'closed and bounded' rather than 'compact'. But actually the Euclidean or Heine-Borel aspect is not crucial here: Cantor's Theorem is really about compactness. We have defined compactness in terms of every open covering having a finite subcovering. One can define compactness equivalently in terms of the *finite intersection property*: every finite subfamily having a non-empty intersection. We quote: compactness is equivalent to every family of closed sets with the finite intersection property having a non-empty intersection. See e.g. Th. 1 of
J. L. KELLEY, *General Topology*, Van Nostrand, 1955; Ch. 5, Compact spaces.
The finite intersection property is automatically satisfied for decreasing sequences of non-empty sets, as here.

Theorem (HEINE). If f is a continuous function on a compact set K, f is uniformly continuous on K.

Proof. Choose $\epsilon > 0$. As f is continuous on K, for each $x \in K$ there exists $\delta(x) > 0$ such that for all $y \in K$ with $|y - x| < \delta(x)$, $|f(y) - f(x)| < \epsilon/2$.

Let $J(x) = N(x, \delta(x)/2)$ be the open disc with centre x and radius $\delta(x)/2$. Then $\{J(x) : x \in K\}$ is an open covering of K. By compactness, there is a finite subcovering, $\{J(x_1), \ldots, J(x_n)\}$ say. Write $\delta := \frac{1}{2}\min\{\delta(x_1), \ldots, \delta(x_n)\}$. Then $\delta > 0$ (minimum of a *finite* family of positive numbers is positive).

If $x, y \in K$ with $|x - y| < \delta$, then $x \in J(x_i)$ for some i (as the $J(x_i)$ cover K). So $|x - x_i| < \frac{1}{2}\delta(x_i) < \delta(x_i)$. So

$$|f(x) - f(x_i)| < \epsilon/2. \qquad (i)$$

Also $|y - x_i| \leq |y - x| + |x - x_i| < \delta + \frac{1}{2}\delta(x_i) \leq \delta(x_i)$. So

$$|f(y) - f(x_i)| < \epsilon/2. \qquad (ii)$$

By (i) and (ii),

$$|f(y) - f(x)| \leq |f(y) - f(x_i)| + |f(x_i) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

So f is uniformly continuous on K. //

Corollary. If $f : [a, b] \to \mathbf{R}$ is continuous, f is uniformly continuous on $[a, b]$.

Proof. $[a, b]$ is closed and bounded, so compact by the Heine-Borel Theorem. The result follows by Heine's Theorem above. //

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