M2PM3 COMPLEX ANALYSIS: ASSESSED COURSEWORK 2, 2010

Set Mon 22.2.2010, Deadline 2pm Wed 3 March 2010; 20 marks

Q1 [**4**].

For $f(z)$ $(z = x + iy)$ regarded as a function of x and y, write

$$\frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

Show that
(i) [**1**] for f holomorphic in a domain D, these partial derivatives exist (so the above are well-defined);
(ii) [**1**] $\partial f/\partial z = f'$;
(iii) [**1**] $\partial f/\partial \bar{z} = 0$.
(iv) [**1**] If f has continuous partials and $\partial f/\partial \bar{z} = 0$, show that f is holomorphic.

Q2 [**3**] (*Poisson kernel*).

Let $z = re^{i\theta}$, $w = Re^{i\phi}$, $z \neq w$. Show that
(i) [**1**]

$$Re\Big(\frac{w+z}{w-z}\Big) = \frac{|w|^2 - |z|^2}{|w-z|^2};$$

(ii) [**1**] $|w - z|^2 = R^2 - 2Rr\cos(\theta - \phi) + r^2$;
(iii) [**1**]

$$Re\Big(\frac{w+z}{w-z}\Big) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}.$$

Q3 [**5**] (*Poisson integral*).

Let f be holomorphic on the closed disc $\bar{D}(0, R) := \{z : |z| \leq R\}$, $z = re^{i\theta} \in D(0, R) = \{z : |z| <\}$. By applying the Cauchy integral formula to fg, where $g(w) := (R^2 - r^2)/(R^2 - w\bar{z})$, or otherwise, show that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}.f(Re^{i\phi})d\phi.$$

Q4 [**4**].

For $C(0, 1)$ the unit circle, show that

$$\int_{C(0,1)} cosec^2 z dz = 0.$$

Q5 [**4**].

Show that

$$\int_{C(0,1)} (Im\ z)^2 dz = 0.$$

Note. Cauchy's Theorem does not apply in either of Questions 4 or 5 – and you should say why not.

NHB