M2PM3 COMPLEX ANALYSIS: SOLUTIONS TO ASSESSED COURSEWORK 2, 2010

4.3.2010 20 marks

Q1 [**4**].
(i) [**1**] For f holomorphic, $f = u + iv$, u and v are differentiable w.r.t. x and y (as in lectures: for $\partial/\partial x$, take the difference $z - z_0$ real; for $\partial/\partial y$, take it imaginary).
(ii) [**1**] $f_x = u_x + iv_x$, $f_y = u_y + iv_y$, so

$$\partial f/\partial z := \frac{1}{2}(f_x - if_y) = \frac{1}{2}[(u_x + iv_x) - i(u_y + iv_y)] = \frac{1}{2}(u_x + v_y) + \frac{1}{2}i(v_x - u_y).$$

By the Cauchy-Riemann equations, this is $u_x + iv_x$. As in Lecture 12 (5.2.2010), this is $f'(z)$.
(iii) [**1**]

$$\partial f/\partial \bar{z} := \frac{1}{2}(f_x + if_y) = \frac{1}{2}[(u_x + iv_x) + i(u_y + iv_y)] = \frac{1}{2}(u_x - v_y) + \frac{1}{2}i(v_x + u_y).$$

By the Cauchy-Riemann equations, this is 0.
(iv) [**1**] As above in (iii), $\partial f/\partial \bar{z} = 0$ is equivalent to the Cauchy-Riemann equations. This and continuity of partials gives differentiability, i.e. holomorphy, as in lectures.

Q2 (*Poisson kernel*) [**3**].
(i) [**1**] $(w - z)(\bar{w} - \bar{z}) = (w - z)\overline{(w - z)} = |w - z|^2$. Also

$$(w+z)(\bar{w}-\bar{z}) = w\bar{w} - w\bar{z} + z\bar{w} - z\bar{z} = |w|^2 - |z|^2 - ((w\bar{z}) - \overline{(w\bar{z})}) = |w|^2 - |z|^2 - 2iIm(w\bar{z}).$$

So multiplying top and bottom by $\bar{w} - \bar{z}$,

$$\frac{w+z}{w-z} = \frac{|w|^2 - |z|^2 - 2iIm(w\bar{z})}{|w-z|^2},$$

and the result follows on taking real parts.
(ii) [**1**] $|w - z|^2 = (w - z)(\bar{w} - \bar{z}) = w\bar{w} - w\bar{z} - \bar{w}z + z\bar{z} = |w|^2 - [(w\bar{z}) + \overline{(w\bar{z})}] + |z|^2$

$= R^2 - 2Re(Re^{i\phi}.re^{-i\theta}) + r^2 = R^2 - 2Rr\cos(\theta - \phi) + r^2$.
(iii) [**1**] Combining,

$$Re\Big(\frac{w+z}{w-z}\Big) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2}.$$

Q3 (*Poisson integral*) [**5**].

$g(w)$ is holomorphic except where $w = R^2/\bar{z} = (R^2/r)e^{-i\theta}$, which is outside D as $r < R$. So as f is holomorphic in D, so is fg. So by Cauchy's Integral Formula [**1**],

$$f(z)g(z) = \frac{1}{2\pi i}\int_{C(0,R)} \frac{f(w)g(w)}{w-z}dw.$$

But $g(z) = (R^2 - r^2)/(R^2 - z\bar{z}) = (R^2 - r^2)/(R^2 - |z|^2) = 1$ as $|z| = r$. So this gives [**1**]

$$f(z) = \frac{R^2 - r^2}{2\pi i}\int_{C(0,R)} \frac{f(w)}{(w-z)(R^2 - w\bar{z})}dw.$$

The right is [**1**]

$$\frac{R^2 - r^2}{2\pi i}\int_0^{2\pi} \frac{f(Re^{i\phi})}{(Re^{i\phi} - re^{i\theta})(R^2 - Rre^{i(\phi-\theta)})}.iRe^{i\phi}d\phi,$$

or [**1**]

$$\frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{i\phi})}{(R - re^{i(\theta-\phi)})(R - re^{i(\phi-\theta)})}.d\phi.$$

So [**1**]

$$f(re^{i\theta}) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{f(Re^{i\phi})}{(R^2 - 2Rr\cos(\theta-\phi) + r^2)}.d\phi.$$

Note. We now know that harmonic functions u are exactly the real parts of holomorphic functions f. So taking real parts of $f = u + iv$:

$$u(re^{i\theta}) = \frac{R^2 - r^2}{2\pi i}\int_0^{2\pi} \frac{u(Re^{i\phi})}{(R^2 - 2Rr\cos(\theta-\phi) + r^2)}.d\phi.$$

These are the *Poisson integral formulae*, giving a holomorphic or harmonic function inside a disc in terms of an integral involving its values on the boundary.

Q4 [**4**].

$d(\cot z)/dz = cosec^2 z$ [**1**]. So as the unit circle is closed,

$$\int_{C(0,1)} cosec^2 z dz = \int_{C(0,1)} \frac{d}{dz} \cot z dz = \int_{C(0,1)} d \cot z = [\cot z]_{C(0,1)} = 0,$$

by the Fundamental Theorem of Calculus [**2**]. Cauchy's Theorem does *not* apply, as $cosec^2 z$ has a singularity at 0 (a double pole) [**1**]. [Cauchy's Residue Theorem does apply (the residue is 0 as the pole is double rather than single) – but the lecture for this is after the deadline!]

Q5 [**4**].

Parametrize $C(0,1)$ by $e^{i\theta}$, $0 \leq \theta \leq 2\pi$. For $f(z) = (Im\ z)^2$, $z = e^{i\theta}$, $f(z) = \sin^2 \theta$ [**1**], so the integral is

$$I = \int_0^{2\pi} \sin^2 \theta . ie^{i\theta} d\theta = -\int_0^{2\pi} \sin^3 \theta d\theta + i \int_0^{2\pi} \cos \theta \sin^2 \theta dt = I_1 + iI_2,$$

say [**1**].

$$I_1 = \int_0^{2\pi} (1 - \cos^2 \theta) d \cos \theta = [\cos \theta - \frac{1}{3} \cos^3 \theta]_0^{2\pi} = 0,$$

by periodicity of cos. Similarly,

$$I_2 = \int_0^{2\pi} \sin^2 \theta d \sin \theta = \frac{1}{3} [\sin^2 \theta]_0^{2\pi} = 0. \qquad [\mathbf{1}]$$

Cauchy's Theorem does not apply since the function $Im\ z$ is not holomorphic [**1**].

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