m2pm3l10.tex

Lecture 10. 1.2.2010.

10. *Termwise differentiation and integration.*

If $\sum u_n(x)$ is a convergent series of functions and $\int \{\sum u_n(x)\,dx\} = \sum [\int u_n(x)\,dx]$, we say $\sum u_n(x)$ can be *integrated term-by-term*, or *termwise*. If $\{\sum u_n(x)\}' = \sum u_n'(x)$, we say $\sum a_n(x)$ can be *differentiated termwise*. We quote:

(I) If $\sum u_n(x)$ converges *uniformly*, it can be integrated termwise.

(D) If $\sum u_n'(x)$ *converges uniformly*, then $\sum u_n(x)$ can be differentiated termwise.

For a power series $\sum a_n z^n$, we get $\sum n a_n z^{n-1}$ by differentiating termwise; similarly we get $\sum a_n z^{n+1}/(n+1)$ by integrating termwise. All three power series have the same R of C (the shift of suffix from n to $n \pm 1$ makes no difference, and neither do the factors of n or $n+1$, as $n^{1/n} = e^{(\log n)/n} \to e^0 = 1$). Combining:

Theorem. A power series can be differentiated (or integrated) termwise inside its circle of convergence.

We can do this arbitrarily often ('infinitely often'):

Theorem. A power series can be differentiated (termwise) *infinitely often* inside its circle of convergence.

We shall see later (Cauchy-Taylor Theorem, II.7) that the functions we study in Chapter II - *holomorphic functions*, 'differentiable once', are exactly those representable by power series. So:

$$f \text{ differentiable } \textit{once} \Leftrightarrow f \text{ differentiable infinitely often.}$$

This is a total contrast to Real Analysis.

Chapter II. Holomorphic (Analytic) Functions: Theory

1. Special Complex Functions.

1. *Polynomials.* $f(z) = a_0 + a_1 z + ... + a_n z^n$ $(a_i \in \mathbf{C}, a_n \neq 0)$. This is a (complex) *polynomial* of *degree* n. We shall prove (II.6, Fundamental Theorem of Algebra) that f has n roots.

2. *Exponentials*
$\exp(z) = \sum_{n=0}^{\infty} z^n/n!$. This function is absolutely and uniformly convergent on all closed discs in $\mathbf{C}$.

$$\exp(z_1 + z_2) = \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} z_1^k z_2^{n-k} \binom{n}{k}.$$

Here $0 \leq k < \infty$. Putting $l := n - k$: $0 \leq k, l < \infty$, giving

$$\exp(z_1 + z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{z_1^k}{k!} \times \frac{z_2^l}{l!} = \left(\sum_{k=0}^{\infty} \frac{z_1^k}{k!} \right) \left(\sum_{l=0}^{\infty} \frac{z_2^l}{l!} \right) = \exp(z_1) \times \exp(z_2)$$

(the rearrangement is justified by absolute convergence). That is,

$$\exp(z_1 + z_2) = \exp(z_1) \times \exp(z_2).$$

Recall $e = \exp(1) = \sum_{n=0}^{\infty} 1/n!$. Now:

$$\exp(nz) = [\exp(z)]^n \quad (n \in \mathbf{N}); \qquad \exp(z/m) = \exp(z)^{1/m} \quad (n \in \mathbf{N}).$$

Combining:

$$\exp(\frac{m}{n} z) = [\exp(z)]^{m/n} \quad (m, n \in \mathbf{N}); \qquad \exp(qz) = [\exp(z)]^q \quad (q \in \mathbf{Q}).$$

Taking $z = 1$ gives $\exp(q) = [\exp(1)]^q = e^q$ for $q \in \mathbf{Q}$. So $\exp(q) = e^q$ for $q \in \mathbf{Q}$. Hence

$$\exp(x) = e^x \qquad (x \in \mathbf{R})$$

(both sides are continuous: take q_n rational, $q_n \to x$.

3. *Trigonometric functions.* Recall from Chapter I:

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ..., \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - ..., \\ \exp(iz) &= \cos(z) + i \sin(z). \end{aligned}$$

For $z = \theta$ real, $\exp(i\theta) = \cos\theta + i \sin\theta$ (Euler's formula, Chapter I).
We *define* $e^{i\theta}$ by $e^{i\theta} = \exp(i\theta) = \cos\theta + i \sin\theta$. Then for $z = x + iy$,

$$\exp(z) = \exp(x + iy) = \exp(x). \exp(iy) = e^x.e^{iy},$$

by above. We *define* e^{x+iy}, or e^z, as the RHS ("rule of indices", complex case). Then

$$\exp(z) = e^z \quad (z \in \mathbf{C}).$$

Henceforth, we use e^z for $\exp(z)$.