m2pm3l11.tex

Lecture 11. 4.2.2010.

4. *Hyperbolic functions*

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \qquad \frac{d}{dz}\cosh z = \sinh z,$$
$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \qquad \frac{d}{dz}\cosh z = \cosh z,$$
$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \cosh^2 z - \sinh^2 z = 1.$$

5. *Logarithms*

Recall that in Real Analysis, log is the inverse function of exp:

$$\log x = y \text{ means } e^y = x.$$

This extends to $\mathbf{C}$, as follows: for $z, w \in \mathbf{C}$,

$$\log z = w \text{ means } e^w = z.$$

But: $e^{2\pi i} = 1$, so $e^{2\pi k i} = 1\ \forall k \in \mathbf{Z}$. So if $e^w = z$, also $e^{w+2\pi k i} = z$. So if $\log z = w$, also $\log z = w + 2\pi k i$: the log is *not* single-valued, and is determined only to within additive multiples of $2\pi i$. In particular, log is not a *function* as previously defined.

There are three ways to proceed:

(i) *Many-valued functions.*

We can regard log as a *many-valued function* (as with $\sin^{-1} = \arcsin$).

(ii) *Cuts.*

Cut the complex plane $\mathbf{C}$ by removing (e.g.) the negative real axis.

(iii) *Riemann surfaces*

Think of log, not going from $\mathbf{C}$ to $\mathbf{C}$, but from R to $\mathbf{C}$, where R is a doubly infinite stack of copies $\mathbf{C}_k$ of $\mathbf{C}$, one for each $k \in \mathbf{Z}$, 'spliced together' along their positive real axes so that $\theta \to \theta + 2\pi$ takes one from $\mathbf{C}_k$ to $\mathbf{C}_{k+1}$. This is a *Riemann surfaces* (G.F.B. RIEMANN (1926-66) in 1851; Felix KLEIN (1849-1925) in 1882).

The origin O is a 'point of bad behaviour' of $\log z$: a *singularity* (see later) – a *branch-point.*

If $e^{w_i} = z_i$ $(i = 1, 2)$:

$$e^{w_1+w_2} = e^{w_1}e^{w_2} = z_1 \cdot z_2,$$

$$w_1 + w_2 = \log(z_1 \cdot z_2),$$

$$\log(z_1 \cdot z_2) = \log z_1 + \log z_2,$$

$$\log(z_1/z_2) = \log z_1 - \log z_2.$$

Recall: in the real case, $y = \log x$ if $e^y = x$. Differentiating (implicitly) w.r.t. x:

$$e^y dy/dx = 1, \qquad dy/dx = 1/e^y = 1/x, \qquad dy = dx/x, \qquad y = \int dx/x.$$

As $e^0 = 0$, $\log 1 = 0$: $y = \int_1^x du/u$; and $y = \log x$:

$$\log x = \int_1^x du/u.$$

Using complex differentiation (II.2), and complex integration (II.4), we can extend this to $\mathbf{C}$, *provided*:
(i) We integrate along the *line-segment* $[1, z]$ joining 1 to z in $\mathbf{C}$;
(ii) $[1, z]$ *avoids* the singularity $z = 0$ (*branch-point*).
So in $\mathbf{C}$,

$$\log z = \int_1^z dw/w, \qquad \text{or} \qquad \int_{[1,z]} dw/w$$

works, *provided* that z does not lie on the negative real line or O, i.e. *provided* that we work with the *cut* plane. For details, see Exam, 2009, Q2.

6. *Complex Powers*
Recall in the real case, $a^x = e^{x \log a}$. In the complex case, $\log z$ is many-valued, so z^w is many valued:

$$z^w := e^{w \log z}.$$

Branch points (continued).

The many-valuedness of $\log z$, z^w arises because if we perform a complete revolution about the origin, the argument arg z increases by 2π. If we prevent complete revolutions about the origin by cutting the plane as above, $\log z$, z^w become single-valued. Here 0 is a *branch-point.* It is the point where different branches of the function (sheets of the Riemann surface) meet. Similarly with $\log(z - z_0)$, $(z - z_0)^w$ for complete revolutions about z_0, where z_0 is a branch point. Similarly, 0 is a branch-point for $z^{1/n}$ (n-valued; the n branches meet at 0).