m2pm3l12.tex

Lecture 12. 5.2.2010.

2. Complex Differentiation and the Cauchy-Riemann Equations

Defn. We say $f : \mathbf{C} \to \mathbf{C}$ is *differentiable at* z_0 with *derivative* w, and write $f'(z_0) = w$, if

$$\frac{f(z) - f(z_0)}{z - z_0} \to w \text{ as } z \to z_0 : \qquad f'(z_o) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The point z_0 is that z may tend to z_0 in ANY way – i.e., from ANY direction; the limit has to be the same for all ways of approach. So,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall z \text{ with } |z - z_0| < \delta, \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$

($arg(z - z_0)$ can be anything!). Write $z - z_0 = h = k + il$ (k, l real), $f = u + iv$ (u, v real): $f(z) = u(x, y) + iv(x, y)$.
1. h real ($l = 0$).

$$\frac{u(x_0 + k, y_0) - u(x_0, y_0)}{k} + i \frac{v(x_0 + k, y_0) - v(x_0, y_0)}{k} \to f'(z_0) \quad (k \to 0) :$$

$$u_x(x_0, y_0) + iv_x(x_0, y_0) = f'(z_0), \quad \text{writing } u_x \text{ for } \partial u/\partial x.$$

2. h imaginary ($k = 0$).

$$\frac{u(x_0, y_0 + l) - u(x_0, y_0)}{l} + i \frac{v(x_0, y_0 + l) - v(x_0, y_0)}{il} \to f'(z_0) \quad (l \to 0) :$$

$$-iu_y(x_0, y_0) + v_y(x_0, y_0) = f'(z_0), \quad \text{writing } u_y \text{ for } \partial u/\partial y.$$

Combining, at (x_0, y_0)

$$u_x = v_y, \quad v_x = -u_y$$

These are called the Cauchy-Riemann Equations, C-R.
So differentiability at $(x_0, y_0) \Rightarrow$ C-R at (x_0, y_0): C-R are *necessary* for differentiability. They are *not sufficient.*

Example. $f(z) = \sqrt{|xy|}$ $(z = x + iy = re^{i\theta})$. So $f, u, v \equiv 0$ on both axes. So $u_x, u_y, v_x, v_y \equiv 0$ on both axes and C-R holds at $(0, 0)$. But:

$$\frac{f(z) - f(0)}{z - 0} = \frac{\sqrt{|r\cos\theta \cdot r\sin\theta|}}{re^{i\theta}} = e^{-i\theta}\sqrt{|\cos\theta\sin\theta|}$$

RHS depends on θ, i.e. *how* $z = re^{i\theta} \to 0$: f is *not* differentiable at 0.
But there is a *partial* converse:

Theorem. If $f = u + iv$ and the partial derivatives u_x, u_y, v_x, v_y exist *and are continuous* in a neighbourhood of z_0, and satisfy the C-R equations at z_0, then f is dfferentiable at z_0.

Proof. Take $h = k + i\ell$ so small that $z = z_0 + k$ is in the neighbourhood where partials are continuous; then

$$u(x_0+k, y_0+\ell) - u(x_0, y_0) = [u(x_0+k, y_0+\ell) - u(x_0, y_0+\ell)] + [u(x_0, y_0+\ell) - u(x_0, y_0)].$$

By the Mean Value Theorem (MVT):

$$\begin{aligned} [u(x_0 + k, y_0 + \ell) - u(x_0, y_0 + \ell)]/k &= u_x(x_0 + \theta k, y_0 + \ell) \\ &= u_x(x_0, y_0) + o(1) \quad \text{as } h \to 0. \end{aligned}$$

(here we use the o-notation for the error term:'$o(1)$ as $h \to 0$' means '$\to 0$ as $h \to 0$'), by continuity of the partial u_x. Similarly,

$$\begin{aligned} [u(x_0, y_0 + \ell) - u(x_0, y_0)]/\ell &= u_y(x_0, y_0 + \theta' \ell) \quad \text{for some } \theta' \in (0, 1) \\ &= u_y(x_0, y_0) + o(1) \quad \text{as } h \to 0. \end{aligned}$$

Combining:

$$u(x_0 + k, y_0 + \ell) - u(x_0, y_0) = ku_x(x_0, y_0) + \ell u_y(x_0, y_0) + o(h),$$

where '$o(h)$' means 'smaller order of magnitude then h as $h \to 0$.' This combines two error terms, $o(k)$ and $o(l)$, both $o(h)$ as $h^2 = k^2 + l^2$, $|k| \leq |h|$, $|l| \leq |h|$. Similarly,

$$v(x_0 + k, y_0 + \ell) - v(x_0, y_0) = kv_x(x_0, y_0) + \ell v_y(x_0, y_0) + o(h)$$

. So

$$\begin{aligned} f(z_0 + h) - f(z_0) &= [u(x_0 + k, y_0 + \ell) - u(x_0, y_0)] + i[v(x_0 + k, y_0 + \ell) - v(x_0, y_0)] \\ &= ku_x + \ell u_y + ikv_x + i\ell v_y + o(h). \end{aligned}$$

Replace u_y, v_y on RHS by $-v_x$, u_x, using the C-R equations. The RHS becomes

$$(k+i\ell)u_x + i(k+i\ell)v_x + o(h) \quad \text{by C-R} = h(u_x + iv_x) + o(h).$$

Divide by h:

$$\begin{aligned} f(z_0+h) - f(z_0)/h &= u_x(x_0,y_0) + iv_x(x_0,y_0) + o(1) \\ &\to u_x(x_0,y_0) + iv_x(x_0,y_0) \qquad \text{as } h \to 0. \end{aligned}$$

So $f'(z_0)$ exists and $= u_x(x_0,y_0) + iv_x(x_0,y_0)$. //

Note. There are *three* other ways to write the RHS in the equation above.