m2pm3l13.tex

Lecture 13. 8.2.2010.

Harmonic functions. With u, v as above,

$$\begin{aligned} u_{xx} &= (v_y)_x \qquad \text{(by Cauchy-Riemann)} \\ &= (v_x)_y \qquad \text{(interchanging the order of partial differentiation)} \\ &= (-u_y)_y \qquad \text{(by Cauchy-Riemann) :} \end{aligned}$$

$$u_{xx} + u_{yy} = 0, \quad \text{or} \quad \Delta u = 0,$$

where Δ is the two-dimensional Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and similarly $\Delta v = 0$. We say that u, v are *harmonic functions.*
Note. To justify interchange of the order of partial differentiation, we need Clairault's theorem (Alexis CLAIRAULT (1713-1765) in 1743), and this needs continuity of partials (here, second partials). We shall find later (II.7) that if we assume differentiability (in the sense of II.2) *once* throughout a disc, or more general region (see II.3 below for details), we get differentiability *infinitely often.* We thus have all the smoothness we need, and will prove this later, but it would divert us too much to prove this now, so we assume it for now.

Gradient and Directional Derivative.
The *gradient* of u, grad u or ∇u, is the 2-vector $\begin{pmatrix} u_x \\ u_y \end{pmatrix}$. The *directional derivative* $D_{\mathbf{u}}u$ of u is the *direction* of the unit vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is the scalar

$$D_{\mathbf{u}}u(x,y) = \lim_{t \to 0} \frac{u(x+tu_1, y+tu_2) - u(x,y)}{t} \quad \text{(where this exists).}$$

By the *proof* of the Theorem above: if the partials of u exist and are continuous,
(i) $D_{\mathbf{u}}u$ exists, $\forall \mathbf{u}$;
(ii) $D_{\mathbf{u}}u = u_1 u_x(x,y) + u_2 u_y(x,y)$: $D_{\mathbf{u}}u = \mathbf{u}.\nabla u$. For in the Proof, $h = k + il$; $u_1 \leftrightarrow k$, $u_2 \leftrightarrow l$; $h = k + il \leftrightarrow u_1 + iu_2 \leftrightarrow \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ ('$1 \leftrightarrow \mathbf{i}$, $i \leftrightarrow \mathbf{j}$'). So,

$$\text{If } \mathbf{u} \perp \nabla u, D_{\mathbf{u}}.u = 0.$$

The directional derivative is the rate of change of u in direction $\mathbf{u}$. This is 0 along a tangent to the curve $u = const$. So

$$\text{grad } u \perp \text{(tangent to) } u = const.$$

By Cauchy-Riemann,

$$\nabla u.\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} . \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} . \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = -u_x u_y + u_x u_y = 0 :$$

$$\nabla u \perp \nabla v.$$

Combining: Curves u const. and v const. cut orthogonally.

Electromagnetism (EM).
u const.: equipotential curves (curves of const EM potential),
v const.: lines of force.
Fluid Mechanics.
u const.: equipotentials,
v const.: lines of flow.
Gravitational Potential (OS Maps).
u const.: *contours* (curves of constant height),
v const.: lines of either water flow or steepest ascent/decent.

Harmonic Conjugates Given u, to find v and f. By C-R, $v_x = u_y$. Integrate w.r.t. y:

$$v = \int u_x(x, y)\, dy + F(x).$$

Differentiating w.r.t. x:

$$v_x = \frac{\partial}{\partial x}[\int u_x\, dy] + F^{'}(x).$$

By C-R, $v_x = -u_y$:

$$-u_y = \frac{\partial}{\partial x}[\int u_x\, dy] + F^{'}(x).$$

Hence $F^{'}$, F, v, $f = u + iv$.
The function v is called the *harmonic conjugate* of u, $v = \tilde{u}$, $f = u + iv \to f = u + i\tilde{u}$.
Similarly, given v we can find u.