m2pm3l14.tex

Lecture 14. 11.2.2010.

3. Connectedness.

Connectedness is a *topological property*, not specific to **C**. We meet it now in Chapter II, rather than in I.2, as we did not need it there – here it is essential.

Defn. In a topological space (in particular a metric space, in particular $\mathbf{R}^d$ or **C**) an open set S is *disconnected* $\Leftrightarrow$ it can be expressed as the union of two disjoint non-empty open sets. Otherwise S is *connected*.

Two examples:
1. In **R**, $(-1, 0) \cup (0, 1)$ is disconnected, but $(-1, 1) = (-1, 0) \cup \{0\} \cup (0, 1)$ is connected.

This is the key illustrative example. The left and right open intervals here have a common end-point, but as this is missing and not in the set its absence serves to *disconnect* their union. The next example is the analogue of this in the complex plane.

2. In **C**, $N(-1/2, 1/2) \cup N(1/2, 1/2)$ is disconnected, but $N(-1/2, 1/2) \cup \{0\} \cup N(1/2, 1/2)$ is connected (see Lecture 15 for why – this set is not open, so this does not follow from the above).

Defn. 1. An arbitrary set S is *connected* if and only if it cannot be covered by two open sets whose intersection with S are disjoint and non-empty.
2. A connected set S is *simply connected* iff its complement (in **C**) S^c is connected; otherwise S is *multiply connected*.

Three examples:
1. **Annulus**: $\{z : 1 < |z| < 2\}$ is multiply connected.
2. **Disc**: $\{z : |z| < 1\}$ is simply connected.
3. **Punctured disc**: $\{z : 0 < |z| < 1\}$ is multiply connected.

Connected Sets in **R**.

We quote: the connected sets on **R** are the *intervals*.

This shows that 'connected' as a technical term defined above is being used in a sense consistent with its use in ordinary language.

Open Sets in $\mathbf{R}$. We quote: S open in $\mathbf{R} \Leftrightarrow S$ is a finite union or countably union of open intervals.

Continuity and Connectedness.
Example. In $\mathbf{R}$: $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0) \end{cases}$
(unit jump function, Heaviside function). f is continuous except at 0, where it has a jump discontinuity. Now modify this example by deleting the origin from the domain of definition:
$f : (-\infty, 0) \cup (0, \infty) \to \mathbf{R}$, $f(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x > 0) \end{cases}$.
f is now *continuous.* Its only possible point of discontinuity is no longer there.
In $\mathbf{C}$:
$f : N(-1/2, 1/2) \cup \{0\} \cup N(1/2, 1/2) \to \mathbf{R}$, $f(z) = \begin{cases} 0 & (|z + \frac{1}{2}| < 1/2) \\ 1 & (z = 0 \text{ or } |z - \frac{1}{2}| < 1/2) \end{cases}$.
f is discontinuous at 0.
But if $f : N(-1/2, 1/2) \cup N(1/2, 1/2) \to \mathbf{R}$, $f \equiv 0$ on $N(-1/2, 1/2)$, $f \equiv 1$ on $N(1/2, 1/2)$. f is now *continuous* (indeed, infinitely differentiable). From the point of view of later results, this function has **very bad** behaviour. We will build a theory where, knowing the function values in *any* disc (however small) determines the function values anywhere. Such a theory must *exclude* examples such as the above. We do this by restricting the domain of the definition of a function to be *connected.* The above example then becomes not one function but *two* – one $\equiv 0$ on the left-hand disc, the other $\equiv 1$ on the other disc.

Defn. 1. A *domain* D is a non-empty, open, connected subset of $\mathbf{C}$ (domains are sometimes called *regions.*
These are the sets suitable as *domains of definition* for functions f differentiable in the sense of II.2.
2. A function f is *holomorphic* if it is differentiable (in the sense of II.2) in some domain D: $f : D \longrightarrow \mathbf{C}$. We then say f is *holomorphic in D.*
Note. D is non-empty (so a function can be defined on on it), open (differentiable $\leftrightarrow$ open domain: differentiability at a point requires us to form diference quotients at *all* nearby points; all must be in the domain of definition, which must thus be open) and connected (to exclude examples such as the above).