m2pm3l15.tex

Lecture 15. 12.2.2010.

Defn. A set S is *polygonally connected* if any two points in S can be joined by a polygonal [continuous piecewise-linear curve] that is entirely contained in S.

We quote: In $\mathbf{C}$, an open set S is connected $\Leftrightarrow$ it is polygonally connected.

For Proof, see e.g. Ahlfors, p.56-57 (Chapter 2, Section 1.3).

Note. W.l.o.g., we can take the line-segments of the polygonal horizontal or vertical.

If $z_1, z_2 \in S$, write $z_1 \sim z_2$, (or $z_1 \overset{S}{\sim} z_2$) if z_1, z_2 can be joined by a polygonal path that is contained in S. This is an *equivalence relation* (reflexive, symmetric, and transitive), so it decomposes S into (disjoint) equivalence classes, called the *connected components* of S.

S is connected $\Longleftrightarrow$ it only has *one* connected component.

Recall that a connected set S is simply connected $\Longleftrightarrow$ S^c is connected.

Call S:

doubly connected $\Longleftrightarrow$ S^c has *two* connected components ('one hole' – e.g., an annulus);

triply connected $\Longleftrightarrow$ S^c has *three* connected components ('two holes');

n-ply connected $\Longleftrightarrow$ S^c has n connected components ('$n-1$ holes').

Note. When we meet Cauchy's Residue Theorem (II.7), we find that our functions f holomorphic on domains D have points of bad behaviour – *singularities*. Each singularity needs to be excluded from D (by making a 'hole'): all the action is at the singularities.

4. Paths, Line Integrals, Contours

Defn. A *curve* $\gamma : [a, b] \to \mathbf{C}$ is a C^1-function γ,

$$\gamma : t \mapsto \gamma(t) = \gamma_1(t) + i\gamma_2(t).$$

Call $\gamma(a)$, the *beginning point* or *start* of γ, $\gamma(b)$ the *end-point* or *end* of γ.

If $\gamma : [a, b] \to G$, G open, $G \subset \mathbf{C}$, call γ a curve *in* G.

If $f : G \to \mathbf{C}$ is holomorphic, $f \circ \gamma(t) \mapsto f(\gamma(t)) : [a, b] \to \mathbf{C}$, and

$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t)$ (Chain Rule).

We often need to join curves 'end to end', allowing 'corners', where things are not smooth.

Defn. 1. A *path* γ is a finite set of curves, $\gamma = \{\gamma_1, ..., \gamma_n\}$ (where each γ_i is in C^1), s.t. the end-point of each γ_i is the start of γ_{i+1}.

2. An open set $G \subset \mathbf{C}$ is *arcwise connected* if any 2 points of G can be joined by a path entirely contained in G.

Polygonally connected $\Rightarrow$ arcwise connected (the joining polygonal path is a joining path).

As above: for open sets in $\mathbf{C}$, polygonally connected $\Leftrightarrow$ connected. More is true. We quote: for open sets in $\mathbf{C}$,

$$\text{connected} \Leftrightarrow \text{polygonally connected} \Leftrightarrow \text{arcwise connected.}$$

In a path $\gamma = \{\gamma_1, ...\gamma_n\}$ with γ_i *parametrised* by $[a_i, b_i]$, we may take

$$a = a_1 < b_1 = a_2 < b_2 = a_3 < ... < b_{n-1} = a_n < b_n = b.$$

Then $t \mapsto \gamma(t)$ is C^1, *except* at finitely many points $t = a_{i+1} = b_i$.

Defn. The *path integral*, or *line integral*, $\int_\gamma f$, is

$$\int_\gamma f, \text{ or } \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt := \sum_{i=1}^n \int_{a_i}^{b_i} f(\gamma(t))\gamma'(t)\,dt.$$

Curve Length.

The *length* of a C^1 curve γ on $[a, b]$ is

$$L(\gamma) = \int_a^b \sqrt{\dot\gamma_1^2(t) + \dot\gamma_2^2(t)}\,dt \text{ or } \int_a^b |\dot\gamma_2^2(t)|\,dt.$$

The integrals above are all *Riemann integrals.*

Defn. 1. A path $\gamma : [a, b] \to \mathbf{C}$ is *closed* if $\gamma(b) = \gamma(a)$ (the two end-points of the curve are the same).

2. The path γ is *simple* if $\gamma(s) = \gamma(t)$ only for $s = a$, $t = b$ (no self-intersections). Thus a circle is simple, but a figure of eight is not.

3. A simple closed path γ is called a *contour*. Then $\int_\gamma f$ is called the *contour integral* of f round γ. From now on, we shall be dealing largely with contour integrals.