m2pm3l15.tex
Lecture 16. 15.2.2010.

Lemma (ML Inequality). If $|f| \leq M$ on γ, and γ has length L, then

$$\left|\int_\gamma f\right| \leq ML.$$

Proof. As $|\int .| \leq \int |.|$,

$$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\dot{\gamma}(t)\,dt\right| \leq \int_a^b |f(\gamma(t))|.|\dot{\gamma}(t)|\,dt \leq M\int_a^b |\dot{\gamma}(t)|\,dt = ML. \quad //$$

Example (Fundamental Integral). For C on the unit circle (centre O),

$$\int_C z^n\,dz = \begin{cases} 2\pi i & (n = -1), \\ 0 & (n \neq -1). \end{cases}$$

Proof. Parametrize C by $z = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, $dz = ie^{i\theta}\,d\theta$.

$$\int_C z^n\,dz = \int_0^{2\pi} e^{in\theta}.ie^{i\theta}\,d\theta = i\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i \text{ if } n = -1 \quad (n+1 = 0).$$

But if $n \neq -1$, $n+1 \neq 0$, RHS $= i\left[e^{i(n+1)\theta}/(i(n+1))\right]_0^{2\pi} = 0$, by periodicity of $\cos\theta$, $\sin\theta$, $e^{i\theta}$.

5. Cauchy's Theorem.

Theorem (Cauchy's Theorem for Triangles). If f is holomorphic in a domain D containing a triangle γ and its interior $I(\gamma)$ – then

$$\int_\gamma f = 0.$$

Proof. Join the three midpoints of the sides of the triangle γ. This *quadrisects* γ into 4 *similar* triangles. Call these γ_1 to γ_4: then

$$\int_\gamma f = \sum_1^4 \int_{\gamma_i} f.$$

For, RHS contains 12 terms, 3 for each of the 4 triangles. The 'outer 6' add to $\int_\gamma f$; the 'inner 6' cancel in pairs. So, for at least one i,

$$\left|\int_{\gamma_i} f\right| \geq \frac{1}{4}|I|, \text{ where } I = \int_\gamma f.$$

For if not, each $\left|\int_{\gamma_i} f\right| < \frac{1}{4}|I|$, so

$$|I| = \left|\int_\gamma f\right| = \left|\sum_1^4 \int_{\gamma_i} f\right| \leq \sum_1^4 |\int_{\gamma_i} f| < \sum_1^4 \frac{1}{4}|I| = |I|,$$

a contradiction. W.l.o.g., take this i as 1. So: $\left|\int_{\gamma_1} f\right| \geq \frac{1}{4}|I|$.
Now quadrisect γ_1. Repeating the argument above, at least one of the 4 resulting triangles, γ_2 say, has

$$\left|\int_{\gamma_2} f\right| \geq \frac{1}{4}\left|\int_{\gamma_1} f\right| \geq \frac{1}{4^2}|I|.$$

Continue (or use induction): we obtain a sequence of triangles $\gamma_1, \gamma_2, ..., \gamma_n, ...$ s.t. if Δ denotes the union of γ and its interior $I(\gamma)$ and similarly for Δ_n, γ_n,

$$\Delta_{n+1} \subset \Delta_n \subset ... \subset \Delta_2 \subset \Delta_1 \subset \Delta;$$

lengths: $L(\gamma_n) = 2^{-n}L$ ($L = L(\gamma)$, length of γ); and $4^{-n}|I| \leq \left|\int_{\gamma_n} f\right|$.
The sets Δ_n are decreasing, closed and bounded (so *compact*), and non-empty. So by *Cantor's Theorem* (Handout, or I.2.6),

$$\bigcap_{n=1}^{\infty} \triangle_n \neq \emptyset.$$

Take $z_0 \in \cap_{n=1}^{\infty} \triangle_n$. As Δ is compact, $z_0 \in \Delta$. Since *by assumption* f is holomorphic in D, f is holomorphic at z_0. So

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall z \text{ with } 0 < |z - z_0| < \delta, \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon :$$

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon|z - z_0|.$$

As $diam(\gamma_n) = 2^{-n} \downarrow 0$ as $n \to \infty$, Δ_n tends to $\{z_0\}$ as $n \to \infty$. So

$$\Delta_N \subset N(z_0, \delta)$$

for all large enough n. For this n, and all $z \in \Delta_n$, $|z - z_0| \leq L(\gamma_n) = 2^{-n}L$ (Triangle Lemma, Problems 4). Now

$$f(z_0) + f'(z_0)(z - z_0) = F'(z), \text{ with } F(z) = f(z_0)(z - z_0) + \frac{1}{2}f'(z_0)(z - z_0)^2.$$