m2pm3l17.tex
Lecture 17. 18.2.2010.

Proof of Cauchy's Theorem for Triangles (continued).

$$\begin{array}{rcll}
\int_{\gamma_n} [f(z_0) + f'(z_0)(z - z_0)]\, dz &=& \int_{\gamma_n} F'(z)\, dz & \\
&=& \int_a^b F'(z(t))\dot{z}(t)\, dt & \text{(if } \gamma_n \text{ is parametrised by } [a,b]) \\
&=& [F(z(t))]_{t=a}^b & \text{(Fundamental Th. of Calculus)} \\
&=& F(z(b)) - F(z(a)) & \\
&=& F(z(a)) - F(z(a)) & (z(b) = z(a) \text{ as triangle } \gamma_n \text{ is closed}) \\
&=& 0. &
\end{array}$$

This and and $(*)$ give

$$\left| \int_{\gamma_n} f \right| < \epsilon \int_{\gamma_n} |z - z_0|\, dz.$$

But

$$\begin{array}{rcll}
\int_{\gamma_n} |z - z_0|\, dz &\leq& \max_{\gamma_n} |z - z_0| \cdot L(\gamma_n) & \text{(ML)} \\
&\leq& L(\gamma_n) \cdot L(\gamma_n) & \text{(Triangle Lemma, Problem Sheet)} \\
&\leq& 4^{-n} L^2 & (L(\gamma_n) \leq L \cdot 2^{-n}).
\end{array}$$

Combining:

$$4^{-n}|I| \leq \left| \int_{\gamma_n} f \right| \leq \epsilon \cdot 4^{-n} \cdot L^2. \qquad \text{(ii)}$$

By (i) and (ii):

$$4^{-n}|I| \leq \epsilon 4^{-n} L^2 : \qquad |I| \leq \epsilon \cdot L^2,$$

where L is the length of γ. But $\epsilon > 0$ is arbitrarily small. So $|I| = 0$: $I = 0$: $\int_\gamma f = 0$. //

Cor. (Cauchy's Theorem for Rectangles). If f is holomorphic in a domain D containing a rectangle R and its interior - then $\int_R f = 0$.

Proof. Bisect into two triangles, γ_1, γ_2: $\int_R f = \int_{\gamma_1} f + \int_{\gamma_2} f = 0 + 0 = 0$. //

Similarly one obtains *Cauchy's Therem for Polygons*: Triangulate.

Defn. 1. If $z_1, z_2 \in \mathbf{C}$, write $[z_1, z_2]$ for the *line segment* joining them in $\mathbf{C}$.
2. A domain D is *star-shaped* (or, is a *star domain*) with *star-centre* $z_0 \in D$

if, for all $z \in D$, the line-segment $[z_0, z] \subset D$.

E.g. Discs are star-shaped. Convex sets are star-shaped.
E.g: $D =$ 'Union of two athletics tracks' – star shaped with star-centre z_0, but not convex.

Theorem (of the Antiderivative). If f is holomorphic in a star-domain D with star-centre z_0,

$$F(z) = \int_{[z_0,z]} f,$$

then $F' = f$: F is an antiderivative of f, and f is the derivative of F.

Proof. Take any $z_1 \in D$. We prove $F'(z_1)$ exists and is $f(z_1)$. As D is open and $z_1 \in D$, some neighbourhood $N(z_1, \epsilon_1) \subset D$. For $|h| < \epsilon_1$, $z_1 + h \in D$. As $z_0, z_1, z_1 + d \in D$, the line-segments $[z_0, z_1]$, $[z_0, z_1 + h] \subset D$ (D is star-shaped with star-centre z_0).

Let γ be the triangle with vertices $z_0, z_1, z_1 + h, \Delta$ be the union of γ and its interior $I(\gamma)$. Then $\Delta \subset D$. By Cauchy's Theorem for triangle Δ:

$$\int_\gamma f = 0.$$

That is,

$$\int_{[z_0,z_1]} f + \int_{[z_1,z_1+h]} f + \int_{[z_1,z_1+h,z_0]} f = 0.$$

So

$$F(z_1) + \int_{[z_1,z_1+h]} f - F(z_1 + h) = 0.$$

So

$$\frac{F(z_1 + h) - F(z_1)}{h} = \frac{1}{h} \int_{[z_1,z_1+h]} f. \qquad (i)$$

For c constant,

$$\int_{[z_1,z_1+h]} c = ch : \qquad \frac{1}{h} \int_{[z_1,z_1+h]} c = c. \qquad (ii)$$

As f is continuous (it is holomorphic!),

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |z - z_1| < \delta \Rightarrow |f(z) - f(z_1)| < \epsilon.$$