m2pm3l18.tex
Lecture 18. 19.2.2010.

Proof of the Theorem of the Antiderivative (continued).

For z on the line-segment $[z, z+h]$,

$$|h| < \delta \Rightarrow |z - z_1| < \delta \Rightarrow |f(z) - f(z_1)| < \epsilon.$$

So
$$\left| \int_{[z_1, z_1+h]} \frac{f(z) - f(z_1)}{h} \, dz \right| \leq \frac{1}{|h|}.\epsilon|h| = \epsilon \quad (ML).$$
By (i) and (ii) with $c = f(z_1)$:

$$\frac{F(z_1 + h) - F(z_1)}{k} - f(z_1) = \frac{1}{h} \int_{[z_1, z_1+h]} \{f(z) - f(z_1)\} \, dz.$$

The $|\mathrm{RHS}| < \epsilon$, by above. So $|\mathrm{LHS}| < \epsilon$, $\forall h$ with $|h| < \delta$. So $F^{'}(z_1)$ exists and is $f(z_1)$. //

Theorem (Cauchy's Theorem for Star-Shaped Domains). If D is star-shaped, f is holomorphic in D, and γ is a contour in D, then

$$\int_\gamma f = 0.$$

Proof. By the Theorem of the Antiderivative, f has an antiderivative F: $f = F^{'}$. So

$$\begin{aligned} \textstyle\int_\gamma f &= \textstyle\int_\gamma F^{'} \qquad (f = F^{'}) \\ &= [F]_\gamma \qquad \text{(Fundamental Theorem of Calculus)} \\ &= 0 \qquad (\gamma \text{ is closed}). \ // \end{aligned}$$

Orientation.

By the Jordan Curve Theorem, a contour γ divides the complex plane into two connected components, one bounded, the *inside* $I(\gamma)$ and one unbounded, the *outside*, $O(\gamma)$.

When we traverse the contour γ in the direction of increasing $t \in [a, b]$, if the *interior* $I(\gamma)$ is to the *left*, γ is *positively oriented*. Otherwise γ is *negatively oriented.*

Note. 1. γ is positively oriented unless we say otherwise.
2. We shall restrict to contours γ for which $I(\gamma)$ presents no problems (circles, polygons, etc.) - *or* we assume the Jordan Curve Theorem.

We now assume Cauchy's Theorem for an arbitrary contour γ. We shall need to refer to the interior $I(\gamma)$, and as on the Handout on Cauchy's Theorem, this depends on the Jordan Curve Theorem (JCT). In fact, we shall only need fairly simple contours, such as circles, semicircles and the like, for which the interior can be defined without the JCT. So we could do without an appeal to JCT.

Theorem (Deformation Lemma). If γ_1, γ_2 are contours, $\gamma_1 \subset I(\gamma_2)$ and f is holomorphic in a domain D containing $O(\gamma_1) \cap I(\gamma_2)$ (the region between the two contours) – then

$$\int_{\gamma_2} f = \int_{\gamma_1} f.$$

Proof. As D is connected, it is polygonally connected. Choose points z_1 on γ_1, z_2 on γ_2. We can join z_1, z_2 by a polygonal path $P \subset D$. Now form a contour γ:
(i) Start at z_2 on γ_2.
(ii) Traverse γ_2 (+ve sense) back to z_2.
(iii) Go from z_2 to z_1, along P ('positive sense').
(iv) Traverse γ_1 (-ve sense) back to z_1.
(v) Go from z_1 back to z_2 along P ('negative sense').

$$I(\gamma) = [O(\gamma_1) \cap I(\gamma_2)] \backslash P$$

(this is the region on the left when traversing γ). Then

$$\int_{\gamma} f = 0 \qquad\qquad \text{(Cauchy's Theorem)},$$

i.e.

$$\int_{\gamma_2} f + \int_P f - \int_{\gamma_1} f - \int_P f = 0.$$

So

$$\int_{\gamma_2} f = \int_{\gamma_1} f. \quad //$$