m2pm3l2.tex

Lecture 2. 14.1.2010.

1. Complex Numbers.

Recall $\mathbf{N} := \{1, 2, 3, ...\}$, the set of *natural numbers*. Also, $\mathbf{N}_0 := \{0, 1, 2, ...\} = \mathbf{N} \cup \{0\}$.
We can take these for granted, or proceed as follows:

$$\begin{array}{lcl} 0 & \longleftrightarrow & \emptyset \\ 1 & \longleftrightarrow & \{\emptyset\} \\ 2 & \longleftrightarrow & \{0, 1\} \\ 3 & \longleftrightarrow & \{0, 1, 2\} \end{array}$$

etc. (John von NEUMANN (1903-57) in 1923).
Addition comes with $\mathbf{N}$. Its inverse, subtraction, gives

$$\mathbf{Z} := \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \qquad \text{(integers – Z for Zahl)},$$

an additive group. We can multiply integers, and divide *non-zero* integers, leading to the *rationals*:

$$\mathbf{Q} := \{m/n : m, n \in \mathbf{Z}, n \neq 0\} \qquad \text{(Q for quotient)}.$$

The ancient Greeks had $\mathbf{Z}$ and $\mathbf{Q}$.
We meet the reals $\mathbf{R}$ as:
(i) lengths of line segments (as in Greek geometry);
(ii) infinite decimal expansions.
Constructing $\mathbf{R}$ from $\mathbf{Q}$ is hard, and was not done till 1872, in two ways:
(i) *Dedekind cuts* (or *sections*): Richard DEDEKIND (1831-1916);
(ii) *Cauchy sequences*: Georg CANTOR (1845-1918).
Dedekind cuts are specific to $\mathbf{R}$, as they depend on the *total ordering* of the line ("$x < y, x > y$ or $x = y$"). Cauchy sequences are general, and can be done in any *metric space* [I.2.4].

Argand diagram

Complex numbers $z = x + iy$ correspond to *points* (x, y) in the cartesian plane $\mathbf{R}^2$ or $\mathbf{R} \times \mathbf{R}$, via the *Argand diagram*:

$$z = x + iy \longleftrightarrow (x, y) :$$

Jean-Robert ARGAND (1768-1822) in 1806;
Caspar WESSEL (1745-1818) in 1799 (Danish – translation 1895);
C.F. GAUSS (1777-1855) in 1831.

We call x the *real part* of z and y the *imaginary part*

$$x = Re\ z; \qquad y = Im\ z.$$

Addition:

$$(z_1, z_2) \longrightarrow z_1 + z_2 : \qquad (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Subtraction:

$$(z_1, z_2) \longrightarrow z_1 - z_2 : \qquad (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2).$$

Multiplication:

$$(z_1, z_2) \longrightarrow z_1 z_2 : \qquad (x_1 + iy_1) \times (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

(W.R. HAMILTON (1805-1865) in 1837).

Conjugates and Division

Conjugates. $\overline{z} = x - iy$ is called the (complex) *conjugate* of z.
Note:
1. $\overline{\overline{z}} = z$;
2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;
3. $\overline{z_1 z_2} = \overline{z_2 z_1} = \overline{z_2}\,\overline{z_1}$.

Then
4. $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 > 0$ unless $x = y = 0 \Longleftrightarrow z = 0$.

Note also that

$$x = \frac{1}{2}(z + \overline{z}), \qquad y = \frac{1}{2i}(z - \overline{z}).$$

Division.

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{1}{\|z_2\|^2} z_1 \overline{z_2} = \frac{1}{x_2^2 + y_2^2}(x_1 + iy_1)(x_2 - iy_2) = \frac{x_1 x_2}{x_2^2 + y_2^2} + i\frac{(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \quad (z \neq 0).$$