m2pm3l21.tex

Lecture 21. 26.2.2010.

Theorem (Cauchy's Inequalities). If f is holomorphic and $|f| \leq M$ in $\bar{N}(a,R) = \{z : |z-a| \leq R\}$ – then $|f^{(n)}(a)| \leq n!M/R^n$.

Proof. Take $\gamma = \gamma(a,R)$ in CIF(n). Then by ML,

$$|f^{(n)}(a)| = \frac{n!}{2\pi}\left|\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw\right| = \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(a+Re^{i\theta})iRe^{i\theta}}{R^{n+1}e^{i(n+1)\theta}}\,d\theta\right| \leq \frac{n!}{2\pi R^n}M{\cdot}2\pi = \frac{n!M}{R^n}.$$

7. Cauchy-Taylor Theorem.

Theorem (Cauchy's Form of Taylor's Theorem: Cauchy-Taylor Theorem). If f is holomorphic in $N(a,R)$ $(R>0)$, then there exists constants c_n s.t.

$$f(z) = \sum_{n=0}^\infty c_n(z-a)^n \qquad (z \in N(a,R)),$$

where the coefficients c_n are given by

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad (\gamma = \gamma(a,r),\ 0<r<R).$$

Proof. Choose $z \in N(a,R)$, and then r with $|z-a| < r < R$. By CIF,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dz.$$

But

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\frac{1}{1-\frac{z-a}{w-a}} = \frac{1}{w-a}\cdot\sum_0^\infty \frac{(z-a)^n}{(w-a)^{n+1}}.$$

On $\gamma = \gamma(a,r)$, $|w-a| = r > |z-a|$. So the series converges *uniformly* on w. So we can interchange $\int_\gamma \ldots dw$ and $\sum_0^\infty$:

$$f(z) = \frac{1}{2\pi i}\int_\gamma f(w)\cdot\sum_0^\infty \frac{(z-a)^n}{(w-a)^{n+1}}\,dw = \sum_{n=0}^\infty (z-a)^n\cdot\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw.$$

So (i) $f(z) = \sum_0^\infty c_n(z-a)^n$, where
(ii)$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw = \frac{f^{(n)}(a)}{n!}$, by CIF($n$). //

The following are equivalent:
(i) f is holomorphic (differentiable once) throughout D;
(ii) f is differentiable infinitely often throughout D;
(iii) f is the sum its Taylor series throughout D ('f is *analytic*' in D).
Compare the real case (Lecture 1, $f(x) = \exp(-1/x^2)$)!

Note. Property (iii) – f is *analytic* – is equivalent to (i) – f is holomorphic. So terms analytic and holomorphic can be used interchangeably.

Defn. If $f(z) = \sum_0^\infty c_n(z-a)^n$ and $c_0 = c_1 = ... = c_{k-1} = 0$, but $c_k \neq 0$, i.e. $f(z) = c_k(z-a)^k[1 + c_{k+1}(z-a) + ...]$, then we say that f *has a zero of order k at a*. Then $f(z) = (z-a)^k g(z)$, g holomorphic, $g(a) \neq 0$.

Theorem. If f is holomorphic and not $\equiv 0$, then the zeros of f are *isolated*: each zero a has a neighbourhood containing no zeros other then itself.

Proof. If a is a zerof of f of order k, $f(z) = (z-a)^k g(z)$, g holomorphic, $g(a) \neq 0$. As g is continuous, $g(\cdot) \neq 0$ in some neighbourhood of a. So $f(z) = (z-a)^k g(z) \neq 0$ in this neighbourhood except at a.//

Cor. 1. If f is holomorphic in D, and has zeros z_n with a limit point $z_0 \in D$ – then $f \equiv 0$.

Cor. 2 – Identity Theorem). If f_1, f_2 are holomorphic, and $f_1(z_n) = f_2(z_n)$ at z_n with a limit point $z_0 \in D$ – then $f_1 \equiv f_2$ in D.

Proof. Apply Corollary (1) on $f_1 - f_2$.//

Cor. 3. A holomorphic function is uniquely determined by its values in *any arbitrarily small disc*. Indeed, any infinite set with a limit point in the domain of holomorphic will do.

Note. Recall the example in Section 2.3 Connectedness. The above results *only work* because we have restricted the domains D to be *connected*.

Harmonic functions and holomorphic functions. Call $u(x, y)$ *harmonic* in D, $u \in \mathcal{H}(D)$, if it has continuous 2nd-order partials, and satisfies Laplace's equations: $\Delta u := u_{xx} + u_{yy} = 0$. As in II.2, given u, we can find f holomorphic ($f \in \mathcal{H}(D)$) and $v \in \mathcal{H}(D)$ s.t. $f = u + iv$, $u, v \in \mathcal{H}(D)$.

Recall that in II.2, we saw that continuity of partials of u, v (and the CR equations) was equivalent to holomorphy of f. We now know that this is equivalent to f being infinitely differentiable, and so to u, v being infinitely differentiable. So, for example, assuming continuity of 2nd-order partials (so as to have $u_{xy} = u_{yx}$ by Clairault's Theorem, used in the proof that the CR equations imply u, v harmonic) is in fact no restriction. But we needed the definition of domains D as being connected in II.3, and the subsequent theory above, to establish this.