m2pm3l23.tex

Lecture 23. 4.3.2010.

3. *Analytic continuation by Dirichlet series.*

Example: The Riemann zeta function: $\zeta(s) = \sum_{n=1}\infty 1/n^2$ ($s = \sigma + i\tau$, *Re* $s = \sigma > 1$). Coursework 1: extend $\zeta(s)$ analytically to *Re* $s > 0$. We quote that (using the functional equation for the Riemann zeta function) one can hence continue analytically to $s \in \mathbf{C}$: $\zeta(s)$ holomorphic, except for a singularity at $s = 1$, as $\zeta(1) = \sum 1/n$ diverges (harmonic series).

4. *Analytic continuation by identities.*

Example: The Gamma function. Recall $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$ (Coursework 1). By Real Analysis (Problems 6, Question 3),

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v}\,dv \qquad (0 < x < 1)$$

By Complex Analysis (III.6 below),

$$\int_0^\infty \frac{v^{x-1}}{1+v}\,dv = \frac{\pi}{\sin \pi x} \qquad (0 < x < 1).$$

Combining:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad (0 < x < 1).$$

As $(0,1) \subset \mathbf{C}$ is infinite and contains limit points (all its points are limit points!) we can continue analytically:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \qquad (z \in \mathbf{C}).$$

So RHS has no zeros (or sin would have a pole). So LHS has no zeros:

Cor. $\Gamma(z)$ has no zeros for $z \in \mathbf{C}$ and has poles at $Z = 0, -1, -2, ...$

So $1/\Gamma(z)$ has zeros at $z = 0, -1, -2...$ and no poles. Hence $1/\Gamma(z)$ is entire, and

$$\frac{1}{\Gamma(z)}.\frac{1}{\Gamma(1-z)} = \frac{\sin \pi z}{\pi}.$$

Here $1/\Gamma(z)$ is entire with zeros at $0, -1, -2, ...$, $1/\Gamma(1-z)$ is entire with zeros at $1, 2, ...$, and $\sin \pi z/\pi$ is entire with zeros at the integers.

9. The Maximum Modulus Theorem. See Website - not examinable.

10. Laurent's Theorem and Singularities.

There may be points where f is *not* holomorphic. There, a new kind of expansion is needed (Pierre-Alphonse LAURENT (1813-54), in 1843).

Theorem (Laurent's Theorem). If f is holomorphic in a domain D containing the annulus $0 < R' \leq |z-a| \leq R < \infty$ – then f has an expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \qquad (R' \leq |z-a| \leq R),$$

where

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad (n \in \mathbf{Z})$$

and γ is a positively oriented contour in the annulus surrounding a.

Proof. Write C, C' for the circles centre a radius R, R', γ for the closed path consisting of: (i) C anticlockwise (+ve sense); (ii) a line segment L from C to C'; (iii) C' clockwise (-ve sense); (iv) L reversed, to get back to the starting point on C. The open annulus is the interior $I(\gamma)$ of γ. By CIF,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{C'} \frac{f(w)}{w-z}\, dw$$

(the terms on RHS are from (i) and (iii), since those from (ii) and (iv) cancel).

On C, $|w-a| = R > |z-a|$, so $1/(w-z) = \sum_0^\infty (z-a)^n/(w-a)^{n+1}$ (as in the Proof Cauchy-Taylor Theorem). So

$$\begin{aligned}
\int_C \frac{f(w)}{w-z}\, dw &= \int_C f(w) \sum_0^\infty \frac{(z-a)^n}{(w-a)^{n+1}}\, dw \\
&= \sum_0^\infty \frac{(z-a)^n}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}}\, dw \qquad \text{(by uniform convergence)} \\
&= \sum_0^\infty c_n (z-a)^n, \quad c_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}}\, dw.
\end{aligned}$$

Similarly, on C', $|w-a| = R' < |z-a|$, $1/(z-w) = \sum_0^\infty (w-a)^n/(z-a)^{n+1}$, so

$$-\frac{1}{2\pi i} \int_{C'} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_{C'} \frac{f(w)}{z-w}\, dw = \frac{1}{2\pi i} \int_{C'} f(w) \cdot \sum_{m=0}^\infty \frac{(w-a)^m}{(z-a)^{m+1}}\, dw.$$

Interchanging $\int$ and $\sum$ by uniform convergence as before, this gives

$$\sum_{m=0}^{\infty}(z-a)^{-m-1}\cdot\frac{1}{2\pi i}\int_{C'} f(w)(w-a)^m\,dw = \sum_{n=-\infty}^{-1}(z-a)^n\cdot\frac{1}{2\pi i}\int_{C'}\frac{f(w)}{(w-a)^{n+1}}$$

(writing $n := -m-1$). Combining,

$$f(z) = \sum_{n=\infty}^{\infty} c_n(z-a)^n, \qquad c_n = \frac{1}{2\pi i}\int_{C \text{ or } C'}\frac{f(w)}{(w-a)^{n+1}}.$$

Here, each $\int_{C \text{ or } C'} = \int_\gamma$ Deformation Lemma). //