m2pm3l24.tex
**Lecture 24. 5.3.2010**.

*Example*: $f(z) = \exp(-1/z^2)$, in any annulus $0 < R' \leq |z| \leq R \leq \infty$. This is a *Laurent series* of $f$ at 0.
Recall (Lecture 1) $f$ is *very badly behaved* at 0.

*Defn.* In the Laurent series $\sum_{-\infty}^{\infty} c_n(z-a)^n$ above: $\sum_0^{\infty} c_n(z-a)^n$ (holomorphic, by Cauchy-Taylor) is called the *analytic*, or *regular*, or *holomorphic* part of $f$ at $a$; $\sum_{-\infty}^{-1} c_n(z-a)^n$ is called the *singular* part.

If the singular part is a *polynomial* in $1/(z-a)$ of degree $n$, we say $f$ has a *pole* at $a$ of *order* $n$.
$m = 1$: simple pole, $m = 2$: double pole, etc.
When the singular part is not a polynomial – i.e., $c_n \neq 0$ for *infinitely many* negative $n$ – then $a$ is called an *essential singularity.* Essential singularities are points of very bad behaviour.

*Examples.* 1. $\sin z/z$ at 0: $\sin z/z \to 1$ $(z \to 0)$.
In such situations, if the function is *either* not defined at the point, *or* defined with a value making the function discontinuous there, then the point is a singularity. We then *remove* the singularity by defining the function to have the right value – a *removable singularity.*
2. $\sin(1/z)$ has *zeros* at $1/z = n\pi$, $z = 1/(n\pi)$;
$1/\sin(1/z)$ has *poles* at $z = 1/n\pi \to 0$.
Limits of poles are also points of very bad behaviour, and are also called *essential singularities.*

## 11. Cauchy's Residue Theorem (CRT).

*Defn.* If $f$ has a singularity at $a$, with singular part $\sum_{-\infty}^{-1} c_n(z-a)^n$, the coefficient $c_{-1}$ of $1/(z-a)$ is called the *residue* of $f$ at $a$, $Res_a f$.

**Theorem (Cauchy's Residue Theorem, CRT)**. If $f$ is holomorphic in a domain $D$ except for finitely many singularities $z_i$, and $\gamma$ is a contour with $\gamma \cup I(\gamma) \subset D$ – then

$$\int_\gamma f = 2\pi i \sum Res f,$$

where the sum is over the singularities $z_i$ *inside* $\gamma$.

*Proof.* Surround each singularity $z_i$ inside $\gamma$ by a small circle $\gamma_i$, centre $z_i$, radius $r_i$. Let $\Gamma$ be the closed path consisting of:
(i) $\gamma$, anticlockwise (+ve sense);
(ii) a path $L_i$ from $\gamma$ to $\gamma_i$;
(iii) $\gamma_i$ clockwise (-ve sense);
(iv) $L_i$ reversed back to $\gamma$ (one $\gamma_i$, $L_i$ for each singularity inside $\gamma$).
As in the proof of Laurent's Theorem, $\int_\Gamma f = 0$ (Cauchy's Theorem). But

$$\int_\Gamma f = \int_\gamma f + \sum_i \int_{L_i} f - \sum_i \int_{\gamma_i} f - \sum_i \int_{L_i} f = \int_\gamma - \sum_i \int_{\gamma_i} f.$$

So

$$\int_\gamma f = \sum_i \int_{\gamma_i} f.$$

In $\int_{\gamma_i} f$, the holomorphic part gives 0, by Cauchy's Theorem. If the singular part is $\sum_{-\infty}^{-1} c_n(z-z_i)^n$ (so $c_{-1}$, or $c_{-1,i}$, is the residue at $z_i$),

$$\int_{\gamma_i} f = \int_{\gamma_i} \sum_{-\infty}^{-1} c_n(z-z_i)^n \, dz = \sum_{-\infty}^{-1} c_n \int_{\gamma_i} (z-z_i)^n \, dz$$

(interchanging $\int$ and $\sum$ by *uniform* convergence, as in the Proof of Laurent's Theorem).

By the Fundamental Integral (II.4, Lecture 16), the integral above is $2\pi i$ if $n = -1$ and 0 otherwise. So

$$\int_\gamma f = \sum_i 2\pi i c_{-1} = 2\pi i \sum_i Res f,$$

summed over singularites *inside* $\gamma$. //

*Note on the Exam.* This ends Ch. II: Theory. The exam will consist of:
Q1. Bits and pieces (from anywhere);
Q2,3. Chapter II: Theory (Lectures 10-24);
Q4. Chapter III: Applications (Lectures 25-33) (i),(ii).