m2pm3l25.tex

Lecture 25. 8.3.2010.

Chapter III: Applications

0. Preamble.

Recall (Sixth form, Calculus): differentiation is an automatic process – integration is not. One learns standard methods:
(i) recognising standard forms;
(ii) integration by substitution (*which* one?);
(iii) integration by parts (which way), etc.
Here, we will use CRT to evaluate $\int_\gamma f$. Given an integral to evaluate, our first – and most important – task is to decide: *which* γ?; *which* f?
In what follows, we will learn a variety of standard ways of applying CRT. The exam – Q4 in particular – will involve similar examples.

1. *Integration round the unit circle.* For

$$I := \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta :$$

take γ the unit circle, $z = e^{i\theta}$, $\cos\theta = \frac{1}{2}(e^{i\theta} + ^{-i\theta}) = \frac{1}{2}(z + 1/z)$, $\sin\theta = \frac{1}{2i}(z - 1/z)$, $dz = ie^{i\theta} = iz\, d\theta$ to get $I = \int_\gamma F(z)\, dz$. Evaluate by CRT.
Example 1.

$$I := \int_0^{2\pi} \frac{1}{1 + a\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{1 - a^2}} \qquad (-1 < a < 1).$$

Proof. 1. Real Analysis: Problems 3, Q2 (Weierstrass t-substitution).
2. Complex Analysis:

$$I := \int_\gamma \frac{1}{iz\left(1 + \frac{a}{2}\left(z + \frac{1}{z}\right)\right)}\, dz = -\int_\gamma \frac{1}{\frac{a}{2}(z^2 + 1) + z}\, dz = -\frac{2i}{a}\int_\gamma \frac{1}{z^2 + \frac{2z}{a} + 1}.$$

The integrand is $1/((z - \alpha)(z - \beta))$, where the roots α, β are given by

$$\frac{-\frac{2}{a} \pm \sqrt{\frac{4}{a^2} - 4}}{2} = -\frac{1}{a} \pm \frac{1}{a}\sqrt{1 - a^2} = \frac{1}{a}(-1 \pm \sqrt{1 - a^2}).$$

Only $+$ is inside γ. Integrand: $\frac{1}{(z-\alpha)(z-\beta)}$: Residue at α is $\frac{1}{\alpha - \beta}$ (directly by expanding, or by the Cover-Up Rule, in the next lecture). So

$$Res_{\left(-\frac{1}{a}+\frac{\sqrt{1-a^2}}{a}\right)} \frac{1}{z^2+\frac{2z}{a}+1} = \frac{1}{\frac{-1+\sqrt{1-a^2}}{a} - \left(\frac{-1-\sqrt{1-a^2}}{a}\right)} = \frac{1}{\frac{2\sqrt{1-a^2}}{a}} = \frac{a}{2\sqrt{1-a^2}}.$$

By CRT:

$$I = 2\pi i \left(-\frac{2i}{a}\right) \cdot \frac{a}{2\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}. \qquad //$$

Example 2. For $n = 1, 2, ..., \alpha$ not a multiple of π,

$$\int_0^{\pi} \frac{\cos n\theta - \cos n\alpha}{\cos\theta - \cos\alpha}\, d\theta = \frac{\pi \sin n\alpha}{\sin\alpha} \qquad \text{(Problem 5, Q3).}$$

Proof. Since $\cos(2\pi - \theta) = \cos\theta$, it suffices to prove $I := \int_0^{2\pi} ...\, d\theta = 2\pi \sin n\alpha / \sin\alpha$. Method as above: I is

$$\int_\gamma \frac{(z^n + z^{-n}) - (z_0^n + z_0^{-n})}{(z+\frac{1}{z}) - (z_0 + \frac{1}{z_0})} \frac{dz}{iz} = -i \int \frac{z^n + z^{-n} - z_0^n - z_0^{-n}}{z^n - z(z_0 + \frac{1}{z_0}) + 1}\, dz = -i \int \frac{z^n + z^{-n} - z_0^n - z_0^{-n}}{(z - z_0)(z - \frac{1}{z_0})}\, dz.$$

No singularity at z_0 or $1/z_0$ (zeros in numerator and denominator cancel), but a singularity at 0 from z^{-n}. So (CRT):

$$I = 2\pi i(-i) Res_0 \frac{1}{z^n (z - z_0)(z - z_0^{-1})} = 2\pi \frac{\sin n\alpha}{\sin\alpha}.$$

Note. By Problems 6, Q5:

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2\theta + b^2 \sin^2\theta}\, d\theta = \frac{2\pi}{ab}.$$

Proof: Use γ the ellipse $x^2/a^2 + y^2/b^2 = 1$, $z = x + iy$, $x = a\cos\theta$, $y = b\sin\theta$. By CRT,

$$I := \int_\gamma \frac{dz}{z} = 2\pi i Res_0\ 1/z = 2\pi i.$$

The result now follows by multiplying top and bottom by $\overline{z} = a\cos\theta - ib\sin\theta$ and taking imaginary parts (see Solutions 6 Q5). But this example is geared to the geometry of the *ellipse*. One can do it by the method above – which is geared instead to the geometry of the *circle*. But this is harder (unless $a = b$, when the ellipse is a circle – but then result is trivial and immediate). So *beware*: *think* about your method first, before plunging into calculation!

2. *Translation of the line of integration.*
Recall

$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}.$$

Proof. Real Analysis:

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dxdy$$

$$= \int\int e^{-r^2/2} r\, drd\theta = \int_0^{\infty} e^{-\frac{1}{2}r^2} \cdot r\, dr \int_0^{2\pi} d\theta = 2\pi \int_0^{\infty} e^{-u}\, du = 2\pi. \qquad //$$

Now take $f(z) := e^{-z^2/2}$. This is entire (has no singularities). So for any contour γ, $\int_\gamma f = 0$, by CRT (or, use Cauchy's Theorem). Take γ the rectangle with vertices R, $R+iy$, $-R+iy$, $-R$, with sides γ_1 the interval $[-R, R]$, γ_2 the vertical line from R to $R+iy$, γ_3 the horizontal line from $R+iy$ to $-R+iy$, γ_4 the vertical line from $-R+iy$ to $-R$. So $\sum_1^4 \int_{\gamma_i} f = 0$.
On γ_2, γ_4: $z = \pm R + iuy$ $(0 \le u \le 1)$,

$$f(z) = \exp\{-(\pm R + iuy)^2/2\} = e^{-R^2/2} e^{u^2y^2/2} e^{\pm iRuy} \to 0 \qquad (R \to \infty),$$

as $|e^{\pm iRuy}| = 1$. So $\int_{\gamma_2} f \to 0$, $\int_{\gamma_4} f \to 0$ $(R \to \infty)$. Also

$$\int_{\gamma_1} f \to \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi} \qquad (R \to \infty).$$

Combining,

$$\int_{\gamma_3} f \to \int_{\infty}^{-\infty} e^{-x^2/2}.e^{y^2/2}.e^{-ixy} dx = -\sqrt{2\pi} \qquad (R \to \infty).$$

So (dividing by $\sqrt{2\pi}$ and by $e^{y^2/2}$, and reversing the direction of integration)

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{-ixy} dx = e^{-y^2/2}.$$

The RHS is real, so the LHS is real. Take complex conjugates:

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{ixy} dx = e^{-y^2/2}.$$

This gives the characteristic function (CF) of the standard normal density $\phi(x) := e^{-x^2/2}/\sqrt{2\pi}$, as given in Lecture 1 (the CF is the *Fourier transform* of a probability density).