m2pm3l26.tex

Lecture 26. 11.3.2010.

Example. We give a further example of translation of the line of integration.

$$I := \int_{-\infty}^{\infty} \frac{u^2 e^u}{1+e^{2u}}\, du = \frac{\pi^3}{8}, \qquad \int_{-\infty}^{\infty} \frac{u e^u}{1+e^{2u}}\, du = 0.$$

Take $f(z) := z^2 e^z/(1+e^{2z})$: f has a singularity where $1 + e^{2z} = 0$, $e^{2z} = -1 = e^{(2n+1)i\pi}$, $z_n = (n+1/2)i\pi$. For γ, take the rectangle with vertices $\pm R$, $\pm R + i\pi$, with $\gamma_1 := [-R, R]$, $\gamma_2 := [R, R+i\pi]$, $\gamma_3 := [R+i\pi, -R+i\pi]$, $\gamma_4 := [-R+i\pi, -R]$. The only singularity of f inside γ comes from $n = 0$, at $z = i\pi/2$.

To find the residue of f at this pole, put $z = \frac{i\pi}{2} + \zeta$: as $e^{i\pi/2} = i$, $e^{i\pi} = -1$,

$$f(z) = \frac{(\frac{i\pi}{2}+\zeta)^2 . i . e^{\zeta}}{1-e^{2\zeta}} = \frac{\frac{-i\pi^2}{4}\left(1+\frac{2\zeta}{i\pi}\right)^2 (1+\zeta+\frac{1}{2}\zeta^2+...)}{1-[1+2\zeta+\frac{4}{2}\zeta^2+...]} = \frac{\frac{i\pi^2}{4}\left(1-\frac{4i\zeta}{\pi}...\right)(1+\zeta...)}{2\zeta(1+\zeta+...)}$$

$$= \frac{i\pi^2}{8\zeta}(1-\frac{4i\zeta}{4}...)(1+\zeta...)(1-\zeta...).$$

So $Res_{i\pi/2} f$ = coefficient of $1/\eta$ on RHS $= i\pi^2/8$.
The contributions along γ_2, $\gamma_4 \to 0$, $(R \to \infty)$ (exponentially fast).

$$\int_{\gamma_1} = \int_{-R}^{R} \to \int_{-\infty}^{\infty} \frac{u^2 e^u}{1+e^{2u}}\, du = I.$$

$$\int_{\gamma_3} = \int_{-R}^{R} \frac{(u+i\pi)^2 . (-) e^u}{1+e^{2u}}\, du = \int_{-R}^{R} \frac{(u^2 + 2i\pi u - \pi^2) e^u}{1+e^{2u}}\, du$$

$$\to \int_{-\infty}^{\infty} \frac{u^2 e^u}{1+e^{2u}}\, du + 2i\pi \int_{-\infty}^{\infty} \frac{u e^u}{1+e^{2u}}\, du - \pi^2 \int_{-\infty}^{\infty} \frac{e^u}{1+e^{2u}}\, du.$$

Now

$$\int_{-\infty}^{\infty} \frac{u e^u}{1+e^{2u}}\, du = \int_{-\infty}^{\infty} \frac{u}{e^{-u}+e^u}\, du = 0$$

(odd integrand between symmetrical limits), and (substituting $t := e^u$)

$$\int_{-\infty}^{\infty} \frac{e^u}{1+e^{2u}}\, du = \int_{-\infty}^{\infty} \frac{d(e^u)}{1+e^{2u}} = \int_0^{\infty} \frac{1}{1+t^2}\, dt = [\tan^{-1} t]_0^{\infty} = \frac{\pi}{2}.$$

Combining,

$$\int_{\gamma_3} f = I + 2i\pi.0 - \pi^2.\frac{\pi}{2} = I - \frac{\pi^3}{2}.$$

(The fact that $\int \gamma_3 f$ involves the answer I motivates the choice of the vertices $\pm R + i\pi$. One can with hindsight see this in the form of f, since $(e^{u+i\pi})^2 = e^{2\pi i}.e^{2u} = e^{2u}$.)

By CRT: $\int_\gamma f = 2\pi i Res_{\ i\pi/2}\ f$:

$$I + (I - \frac{\pi^3}{2}) = 2\pi i.i\pi^2/8 = -\frac{\pi^3}{4}: \qquad 2I = \frac{\pi^3}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}: \qquad I = \pi^3/8.//$$

Finding Residues.
Simple Pole of f at α: $f(z) = g(z)/(z-\alpha)$, g holomorphic at α, $g(\alpha) \neq 0$.
Cauchy-Taylor Theorem: $g(z) = \sum_0^\infty c_n(z-\alpha)^n$ ($c_0 \neq 0$). So

$$f(z) = \frac{g(z)}{z-\alpha} = \frac{c_0}{z-\alpha} + c_1 + c_2(z-\alpha) + \ldots :$$

$Res_\alpha f$ = coefficient of $1/(z-\alpha)$ on RHS $= c_0 = g(\alpha)$. So we get the *Cover-up Rule.*

$$Res_\alpha f = g(\alpha) :$$

cover up $1/(z-\alpha)$, then *substitute* $z = \alpha$.
Multiple Poles.
(a) If $z = \alpha$ is a multiple pole, put $z = \alpha + \zeta$ (ζ small), and power-series expand in powers of ζ (as in the Example above). Res_α = coefficient of $1/\zeta$.
(b) *Derivative Rule.*

If α is a pole of f of order m, $f(z) = g(z)/(z-\alpha)^m$ (g holomorphic at α), $g(\alpha) \neq 0$. Cauchy-Taylor Theorem: $g(z) = \sum_0^\infty c_n(z-\alpha)^n$, $c_n = g^{(n)}(\alpha)/n!$. So taking $n = m-1$ gives

$$Res_\alpha f = g^{(m-1)}(\alpha)/(m-1)!$$

Note. In the Example above, the pole is a *simple* one (this is not immediately obvious, but emerges because the Laurent expansion has singular part containing a $1/\zeta$ term only). So we may alternatively find the residue by using the Cover-Up Rule, with

$$f(z) := z^2 e^z/(1+e^{2z}) = g(z)/(z - i\pi/2), \qquad g(z) = \frac{(z - i\pi/2)}{(1+e^{2z})}.z^2 e^z.$$

Using the Cover-Up Rule, we find $Res_{i\pi/2}\ f = g(i\pi/2)$. The z^2e^z factor is $(i\pi/2)^2.e^{i\pi/2} = -i\pi^2/4$ at $i\pi/2$. The fraction on the right is an indeterminate form at $i\pi/2$, which we can evaluate by
(a) L'Hospital's Rule (just as in Real Analysis):

$$\frac{(z - i\pi/2)}{(1 + e^{2z})} \sim \frac{1}{2e^{2z}} \to -1/2 \qquad (z \to i\pi/2),$$

giving the residue as $(-1/2).(-i\pi^2/4) = i\pi^2/8$, as before, or
(b) power-series expansion (along the lines above): with $z = i\pi/2 + \zeta$, the fraction is

$$\frac{\zeta}{[1 - (1 + 2\zeta + \cdots)]} \to -1/2 \qquad (z \to i\pi/2),$$

again as before.

You may wish to compare these two ways of finding the residue in this example. If you have a preference, this may guide you more generally.