m2pm3l27.tex
Lecture 27. 12.3.2010.

3. Infinite Integrals.

We handle these by by a limiting operation.
Example.

$$I := \int_0^\infty \frac{\cos x}{(1+x^2)^2}\, dx = \frac{\pi}{2e}.$$

We prove: $\int_{-\infty}^{\infty} \cos x \; dx/(1+x^2)^2 = 2I = \pi/e$. Take γ the union of $\gamma_1 := [-R, R]$ and γ_2, the closed semi-circle of radius R in the upper half-plane. Take $f(z) = e^{iz}/(1+z^2)^2 = e^{iz}/((z-i)^2(z+i)^2)$ – double pole inside γ at $z = +i$. In the upper half-plane $y \geq 0$, $f(z) = e^{ix}e^{-y}/(1+z^2)^2$, $|f(z)| \leq 1/|(+z^2)^2| = O(1/R^4)$. So by ML,

$$\left|\int_{\gamma_2}\right| = O(1/R^4).\pi R = O(1/R^3) \to 0 \qquad (R \to \infty).$$

$$\int_{\gamma_1} f \to \int_{-\infty}^{\infty} \cos x \; dx/(1+x^2)^2 = 2I \qquad (R \to \infty)$$

(as $\int_{-\infty}^{\infty} \sin x \; dx/(1+x^2)^2 = 0$, odd integrand, symmetrical limits).
By CRT: $\int_\gamma = 2\pi i \; Res_i f$. Near i: $z = i + \zeta$, ζ small.

$$f(z) = \frac{e^{-1}e^{i\zeta}}{[1+(-1+2i\zeta+\zeta^2)]^2} = \frac{e^{-1}e^{i\zeta}}{(2i)^2\zeta^2}.(1+\frac{\zeta}{2i})^{-2} = \frac{1}{4e}\frac{1}{\zeta^2}(1+i\zeta+...)(1+i\zeta+...)$$

$$= -\frac{1}{4e}\frac{1}{\zeta^2}(1+2i\zeta+....):$$

$$Res_i f = \frac{-1}{4e}.2i = -\frac{i}{2e}.$$

By CRT:

$$\int_\gamma f = 2\pi i\left(\frac{2i}{4e}\right) = \frac{\pi}{e}, \qquad \int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \to 2I + 0 = 2I: \qquad 2I = \pi/e. \quad //$$

In the example above, $f(z) = e^{iz}/[(z-i)^2(z+i)^2] = g(z)/(z-i)^2$, where $g(z) := e^{iz}(z+i)^{-2}$.
By the Derivative Rule with $m = 2$, $a = i$:

$$g'(z) = ie^{iz}(z+i)^{-2} + e^{iz}(-2)(z+i)^{-3},$$

$$g'(i) = \frac{ie^{-1}}{(2i)^2} - \frac{2e^{-1}}{(2i)^2} = \frac{-i}{4e} - \frac{1}{4e} = -\frac{i}{2e}.$$

Example (Problems 2 Q5 (ii)).

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{ab(a+b)} \qquad (a,b>0).$$

$$f(z) = \frac{1}{(z+a^2)(z^2+b^2)} = \frac{1}{(z-ia)(z+ia)(z^2+b^2)}$$

So there are poles inside the contour at ib and ia.

$$Res_{ia} f = \frac{1}{2ia(b^2-a^2)}, \qquad \text{and similarly} \qquad Res_{ib} f = -\frac{1}{2ib(b^2-a^2)}.$$

$$\left|\int_{\gamma_2} f\right| = O(1/R^4).O(R) = O(1/R^3) \to 0, \qquad \int_{\gamma_1} f \to I \qquad (R\to\infty).$$

By CRT:

$$I = 2\pi i \sum Res = \frac{2\pi i}{2i}\cdot\frac{1}{b^2-a^2}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{\pi}{ab}\frac{(b-a)}{(b^2-a^2)} = \frac{\pi}{ab(a+b)}.$$

What if $a=b$? We then have *one double pole* at ia inside γ. Evaluate $Res_{ia}f$ by either *series expansion* or *derivative rule* (left as an exercise).

4. Indentation. E.g.

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Take $f(z) = e^{iz}/z$. This has a pole at the origin, which we must exclude from the semi-circular contour we would use as above by a semi-circular indentation round the origin. Take γ the union of γ_1, the semi-circle centre 0 and radius $\epsilon > 0$ in the upper half-plane (clockwise), $\gamma_2 := [\epsilon, R]$, γ_3 the semi-circle radius R in the upper half-plane (anticlockwise) and $\gamma_4 := [-R, -\epsilon]$. By Cauchy's Theorem, $\int_\gamma = 0$. So for $\delta > 0$,

$$|f_{\gamma_3}| = \left|\int_0^{\pi} \frac{e^{i(R\cos\theta + iR\sin\theta)}}{Re^{i\theta}}\cdot iRe^{i\theta}\,d\theta\right| \le \int_0^{\pi} e^{-R\sin\theta}\,d\theta = \int_0^{\delta} + \int_{\delta}^{\pi-\delta} + \int_{\pi-\delta}^{\pi}$$

$$\le \delta + \delta + e^{-R\sin\theta}(\pi - 2\delta): \qquad \limsup_{R\to\infty}\left|\int_{\gamma_3} f\right| \le 2\delta.$$

So as $\delta > 0$ is arbitrarily small: RHS $= 0$. So $\int_{\gamma_3} f \to 0$ $(R\to\infty)$.

$$\int_{\gamma_1} f = \int_0^{\pi} e^{i\epsilon(\cos\theta + i\sin\theta)}\frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\,d\theta = i\int_0^{\pi}(1+O(\epsilon))\,d\theta = i\pi + O(\epsilon) \to i\pi \quad (\epsilon\to 0).$$