m2pm3l28.tex
Lecture 28. 15.3.2010.

$$\left(\int_{\gamma_4}+\int_{\gamma_2}\right) f = \left(\int_{-R}^{-\epsilon}+\int_{\epsilon}^{R}\right)\frac{\cos x + i\sin x}{x}\,dx \to 2i\int_0^\infty \frac{\sin x}{x}\,dx,$$

as $\cos x/x$ is odd, $\sin x/x$ is even and the limits are symetric.
By Cauchy's Theorem: $\int_\gamma f = 0$. Combining:

$$2i\int_0^\infty \frac{\sin}{x}\,dx - i\pi = 0: \qquad \int_0^\infty \frac{\sin}{x}\,dx = \frac{\pi}{2}.$$

Note. f has a simple pole at 0, of residue 1, which would contribute $2\pi i$ if included. The $-i\pi$ above comes from going 'half-way round, the *wrong* way'.

Lemma. As θ increases from 0 to $\pi/2$, $\sin\theta/\theta$ decreases from 1 to $2/\pi$.

Proof: Problems 5, Q2.

Lemma (Jordan's Lemma). If f is meromorphic (no singularities except poles) in the upper half-plane, $y = Imz > 0$, and $|f(z)| \to 0$ ($|z| \to \infty$), uniformly for $\theta = \arg z \in [0, \pi]$, then for $m > 0$ and γ the semi-circle $|z| = R$, $Imz \geq 0$,

$$\int_\gamma e^{inz} f(z)\,dz \to 0 \quad (R \to \infty).$$

Proof.

$$|e^{imz}| = |\exp\{im(R\cos\theta + iR\sin\theta)\}| = \exp\{-mR\sin\theta\} \leq \exp\left\{-\frac{2mR}{\pi}\theta\right\} \quad \text{(Lemma)}.$$

So $\forall\epsilon > 0$, $\exists N$ s.t. if $|z| \geq N$, $|f(z)| < \epsilon$ in upper half-plane. So by ML,

$$\left|\int_\gamma e^{imz} f(z)\,dz\right| \leq \epsilon\int_0^\pi \exp\left\{-\frac{2mR}{\pi}\theta\right\}.R\,d\theta = \epsilon R\cdot\frac{\pi}{2mR}\left[-\exp\left\{-\frac{2mR}{\pi}\theta\right\}\right]_0^\pi$$

$$= \frac{\epsilon\pi}{2m}\left[1 - e^{-2mR}\right] \leq \frac{\epsilon\pi}{2m},$$

which is arbitrary small. //

Note. We could use Jordan's Lemma in

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

(to avoid $\int_0^\pi = \int_0^\delta + \int_\delta^{\pi-\delta} + \int_{\pi-\delta}^\pi$).
Example.

$$\int_{-\infty}^\infty \frac{e^{ixt}}{\pi(1+x^2)}\,dx = e^{-|t|} \quad (t \text{ real})$$

(Lecture 1: characteristic function of Cauchy density).

Proof. Take $\epsilon > 0$. $f(z) = 1/(\pi(1+z^2))$ (to use Jordan's Lemma for $e^{itz}/(\pi(1+z^2)))$. The only singularity inside γ is at $y = i$, a simple pole.

$$Res_i \frac{e^{itz}}{\pi(z-1)(z+1)} = \frac{e^{-t}}{\pi.2i} = \frac{-ie^{-t}}{2\pi}.$$

By CRT:

$$\int_\gamma f = 2\pi i.\left(\frac{-ie^{-t}}{2\pi}\right) = e^{-t}.$$

But

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \to \int_{-\infty}^\infty \frac{e^{i\lambda t}}{\pi(1+x^2)} + 0 \quad \text{(Jordan's Lemma)}.$$

This gives the result for $t > 0$. For $t = 0$, it is a $\arctan^{-1}$ integral. For $t < 0$: replace t by $-t$. //

5. Rotation of the line of integration - Branch points.
Rotation: Use a *sector* as shown.
Branch points. Example: the Gamma function,

$$\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}\,dx \qquad (t > 0).$$