m2pm3l30.tex
Lecture 30. 30.3.2010.

f has a pole at $z = -1 = e^{i\pi}$, of residue $e^{i\pi(a-1)}$. By CRT:

$$I - Ie^{2\pi i(a-1)} = 2\pi i e^{i\pi(a-1)},$$

$$I = \pi.\frac{2ie^{i\pi(a-1)}}{1 - e^{2\pi i(a-1)}} = \pi.\frac{2i}{e^{-i\pi(a-1)} - e^{i\pi(a-1)}}$$

$$= \frac{\pi}{-\sin\pi(a-1)} = \frac{\pi}{\sin\pi(1-a)} = \frac{\pi}{\sin\pi a} = \Gamma(a)\Gamma(1-a). \quad //$$

Note. For $0 < x < 1$, $\Gamma(x)\Gamma(1-x) = \int_0^\infty v^{x-1}dv/(1+v)$. Proof (Solns 6, Q3):

$$\Gamma(x)\Gamma(x-1) = \int_0^\infty t^{x-1}e^{-t}\,dt.\int_0^\infty u^{-x}e^{-u}\,du.$$

Writing $u = tv$ and interchanging the order of integrations, this is

$$\int_0^\infty v^{-x}\,dv\int_0^\infty e^{-t(1+v)}\,dt = \int_0^\infty \frac{v^{-x}}{1+v}\,dv = \int_0^\infty \frac{v^{x-1}}{1+v}\,dv$$

($x \to 1-x$ leaves LHS unchanged, so also RHS).
Branch Points. Recall from Lecture 11:
(i) Complex logs $\log z$ are many-valued;
(ii) hence so are complex powers $z^w := e^{w\log z}$.
This is because the *argument* $\theta = \arg z$ in $z = re^{i\theta}$ is only determined to within integer multiples of 2π. The problem arises from *complete revolutions* about the origin (which we can prevent, e.g. by *cutting* the plane). It is this property of the *origin* that makes it a *branch point.* The branch-point is the point at which the different branches of the function (or, different sheets of the Riemann surface) meet. $\log(z - z_0)$, $(z - z_0)^w$: z_0 is a branch-point, etc.
Complex nth roots of unity.

For $k \in \mathbf{Z}$, $e^{2\pi ik} = 1$. For $n \in \mathbf{N}$, take nth root: $e^{2\pi ik/n} = 1$, $k = 0 \to n-1$. These are the *complex nth roots of unity.* One is real, 1 ($k = 0$). If ω (or ω_n) is an nth root of unity, $\omega^n = 1$:

$$\omega^n - 1 = (\omega - 1)(\omega^{n-1} + \omega^{n-2} + ... + \omega + 1) = 0.$$

So the complex nth roots of unity other than 1 satisfy

$$\omega^{n-1} + \omega^{n-2} + ... + \omega + 1 = 0.$$

The nth roots of unity lie on the unit circle, at the vertices of a regular n-gon with one vertex 1. Draw these for $n = 2, 3, 4, 5, 6$.

7. Summation of Series

Consider the function $\cot \pi z = \cos \pi z / \sin \pi z$, with simple poles at $z = n$. For $z = n + \zeta$, ζ small,

$$\cot \pi z = \frac{\cos(n\pi + \pi\zeta)}{\sin(n\pi + \pi\zeta)} = \frac{(-1)^n \cos \pi\zeta}{(-1)^n \sin \pi\zeta} \sim \frac{1}{\pi\zeta} \quad (\zeta \to 0): \quad Res_n \cot \pi z = \frac{1}{\pi}.$$

Also, *cosec* $\pi z = 1/\sin \pi z$ has simple poles at $z = n$, and

$$Res_n \mathrm{cosec}\ \pi z = \frac{(-1)^n}{\pi}.$$

If $f(z)$ is holomorphic at $z = n$, by the Cover-Up Rule,

$$Res_n\ f(z) \cot \pi z = \frac{f(n)}{\pi}, \quad Res_n\ f(z) \mathrm{cosec}\ \pi z = \frac{(-1)^n f(n)}{\pi}.$$

This suggests a method of summing series $\sum f(z)$ or $\sum (-1)^n f(n)$ by CRT. We need suitable contours.

Lemma. Let C_N be a square with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$. Then cosec πz, $\cot \pi z$ are *uniformly bounded* (in z and N) on C_N.

Proof. Not examinable – see Website.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{(Euler)}.$$

Proof. Take $f(z) = 1/z^2$. Then $f(z) \cot \pi z$ has simple poles at $z = n \neq 0$ residue $f(n)/\pi = 1/(\pi n^2)$, and a triple pole at $z = 0$. Near 0,

$$f(z) \cot \pi z = \frac{\cos \pi z}{z^2 \sin \pi z} = \frac{1 - \frac{\pi^2 z^2}{2} + ...}{z^2 \left(\pi z - \frac{\pi^3 z^3}{6} + ...\right)}$$

$$= \frac{1}{\pi z^3}.(1 - \frac{\pi^2 z^2}{2} + ...)(1 + \frac{\pi^2 z^2}{6} - ...) = \frac{1}{\pi z^3}.(1 - \frac{\pi^2 z^2}{3} + ...).$$

The residue is the coefficient of $1/z$, so

$$Res_0\ f(z) \cot \pi z = -\pi/3.$$