m2pm3l31.tex
Lecture 31. 22.3.2010.

Take $f(z) = 1/z^2$, squares C_N as in the Lemma. By CRT:

$$\left| \int_{C_N} \frac{\cot \pi z}{z^2}\, dz \right| = 2\pi i \sum Res = 2\pi i \left(-\pi/3 + \sum_{n=-N}^{N} \frac{1}{\pi n^2} \right) \quad \text{(Cover-Up Rule)}.$$

By ML: as $\cot \pi z$ is bounded ($= O(1)$) on the C_N, $1/z^2 = O(1/N^2)$ on the C_N, and the C_N have length $O(N)$,

$$\left| \int_{C_N} \frac{\cot \pi z}{z^2}\, dz \right| = O(1).O(1/N^2).O(N) = O(1/N) \to 0 \quad (N \to \infty).$$

Combining, we get

$$-\frac{\pi}{3} + \frac{2}{\pi} \sum_{n=1}^{N} 1/n^2 \to 0 \quad (N \to \infty): \quad \zeta(2) := \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6. \qquad //$$

Similarly,

$$\zeta(4) = \sum_{n=1}^{\infty} 1/n^4 = \pi^4/90.$$

Proof.

$$\frac{\cot \pi z}{z^4} = \frac{\cos \pi z}{z^4 \sin \pi z} = \frac{1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} ...}{z^4 \left(\pi z - \frac{\pi^3 z^3}{6} + \frac{\pi^5 z^5}{120} ... \right)}$$

$$= \frac{1}{\pi z^5} \left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} ... \right) .(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{120} ...)^{-1}$$

$$= \frac{1}{\pi z^5} \left(1 - \frac{\pi^2 z^2}{2} + \frac{\pi^4 z^4}{24} ... \right) .(1 - \{...\})^{-1},$$

say. Since $(1 - \{...\})^{-1} = 1 + \{...\} + \{...\}^2 + ...$, the last factor on the RHS is

$$1 + \frac{\pi^2 z^2}{6} - \frac{\pi^4 z^4}{120} + \frac{\pi^4 z^4}{36} + ... = 1 + \frac{\pi^2 z^2}{6} + \pi^4 z^4 (\frac{1}{36} - \frac{1}{120}) = 1 + \frac{\pi^2 z^2}{6} + \pi^4 z^4 \frac{7}{360},$$

neglecting terms beyond z^4. So multiplying up the last two brackets on the RHS, we get three terms in z^4, each of which will contribute to the residue (coefficient of $1/z$), in view of the z^{-5} factor. This gives

$$Res_0 \frac{\cot \pi z}{z^4} = \text{ coefficients of } 1/z \text{ on RHS } = \frac{1}{\pi}.\pi^4 \left(\frac{7}{360} + \frac{1}{24} - \frac{1}{12} \right) = ... = -\frac{\pi^3}{45}.$$

As before:

$$-\frac{\pi^3}{45} + \frac{2}{\pi}\sum_{n=1}^{N} 1/\pi^4 \to 0 \quad (N \to \infty): \quad \zeta(4) = \sum_{1}^{\infty} 1/n^4 = \pi^4/90. \qquad //$$

8. Expansion of a meromorphic function.
Example.

$$f(z) = cosec\ z - \frac{1}{z} = \frac{1}{\sin z} - \frac{1}{z}.$$

Simple pole at $z = n\pi$, $N \neq 0$. For $z = n\pi + \zeta$,

$$f(z) = \frac{1}{\sin(n\pi + \zeta)} - \frac{1}{n\pi + \zeta} = \frac{(-1)^n}{\sin\zeta} - \frac{1}{n\pi+\zeta} \sim \frac{(-1)^n}{\zeta} \quad (\zeta \to 0):$$

$$Res_{n\pi} f = (-1)^n, \quad n \neq 0.$$

At $n = 0$,

$$f(z) = \frac{z - \sin z}{z \sin z} = \frac{z - \left(z - \frac{z^3}{6}\ldots\right)}{z\left(z - \frac{\pi^3}{6}\ldots\right)} = \frac{\frac{1}{6}z^3 + \ldots}{z^2\left(1 - \frac{z^2}{6}\ldots\right)} \sim \frac{z}{6} \to 0 \quad (z \to 0):$$

no singularity, so no residue.

Recall (III.7 – see Lecture 30 and Handout) that as $cosec\ \pi z$ is bounded on the squares C_N with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$, $cosec\ z$ is bounded on the squares Γ_N with vertices $(N + \frac{1}{2})\pi(\pm 1 \pm i)$. Consider

$$I_N(z) := \int_{\Gamma_N} \frac{f(w)}{w(w-z)}\, dw = \int_{\Gamma_N} \frac{cosec\ w - \frac{1}{w}}{w(w-z)}\, dw.$$

$$\left|\int_{\Gamma_N} \frac{cosec\ w}{w(w-z)}\, dw\right| = O(1).O(1/N^2).O(N) = O(1/N) \to 0 \quad (N \to \infty)$$

$$\left|\int_{\Gamma_N} \frac{\frac{1}{w}}{w(w-z)}\, dw\right| = O(1/N).O(1/N^2).O(N) = O(1/N^2) \to 0.$$

So $I_N(z) \to 0$ as $N \to \infty$. By CRT:

$$I_N(z) = 2\pi i \sum Res \frac{cosec\ w - 1/w}{w(w-z)},$$

over the singularites inside Γ_N. For fixed z, z is *inside* Γ_N for large enough N. By the Cover-Up rule:

$$Res_z = \frac{cosec\ z - 1/z}{z}, \qquad Res_0 = \frac{cosec\ w - 1/w}{w(w-z)} = 0.$$

For $n \neq 0$,

$$Res_{n\pi} = \frac{(-1)^n}{n\pi(n\pi - z)} = \frac{(-1)^n}{(-z)}\left(\frac{1}{n\pi} - \frac{1}{n\pi - z}\right) = -\frac{(-1)^n}{z}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right).$$

So

$$I_N(z) = 2\pi i \left\{ -\sum_{n=-N\, n\neq 0}^{N} \left[\frac{(-1)^n}{z}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)\right] + \frac{cosec\ z - 1/z}{z} \right\} \to 0.$$

We can cancel $2\pi i/z$. Then replace $\sum_{n=-N\, n\neq 0}^{N} = \sum_{-N}^{-1} + \sum_{1}^{N}$ by $\sum_1^N\{...+...\}$: the $1/(n\pi)$ and $-1/(n\pi)$ cancel, and

$$\frac{1}{z - n\pi} + \frac{1}{z + n\pi} = \frac{2z}{z^2 - n^2\pi^2}.$$

We obtain

$$cosec\ z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{(-)^n}{z^2 - n^2\pi^2} = \frac{1}{z} + 2z\sum_{even} - 2z\sum_{odd}. \qquad (i)$$

Similarly,

$$cot\ z = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2\pi^2}. \qquad (ii)$$

In Lecture 32 (lost because of the end of term), we start with (i) and (ii), integrate, and obtain the infinite products for sin, cos and tan. These give extensions to entire functions of the Fundamental Theorem of Algebra, displaying a polynomial as a product of linear factors vanishing at its roots. From these, we can obtain $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and Wallis's product for π (again).