m2pm3l33.tex

Lecture 33. 25.3.2010

Contours for Chapter 3.

1. Circles (III.1, Lecture 25);
2. Ellipses (III.1, Lecture 25, Problems 6 Q6);
3. Squares (III.7, III.8, Lecture 30, 31);
4. Rectangles (III.2, Lecture 25, 26);
5. Semicircles (III.3, Lecture 27);
6. Sectors (III.9, Lecture 33);
7. Indented semicircle, (III.4, Lecture 27);
8. Indented sector (III.5, Lecture 28-29);
9. Keyhole contour (III.6, Lecture 29).

9. Further examples.

1. Consider

$$I_n := \int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi}{n \sin \pi/n} \qquad (n = 2, 3, ...).$$

First Proof: sector contour. Let $f(z) := 1/(1+z^n)$, and take the contour as a sector, with $\gamma_1 = [0, R]$, γ_2 on the arc $|z| = R$, $0 \le \arg\ z \le 2\pi/n$, and γ_3 the path back to the origin.

By ML, $\int_{\gamma_2} f = O(R^{-n}.R) \to 0$ as $R \to \infty$.

$$\int_{\gamma_1} f \to I \qquad (R \to \infty).$$

On γ_3, z goes from R to O, $z = xe^{2\pi i/n}$, $dz = e^{2\pi i/n}\,dx$. So $\int_{\gamma_3} f \to -e^{2\pi i/n} I$. So

$$\int_\gamma f \to I\left(1 - e^{2\pi i/n}\right).$$

By CRT:

$$\int_\gamma f = 2\pi i \sum Res f,$$

and f singular where $z^n = -1 = e^{i\pi} = e^{(2k+1)\pi}$, $z = e^{i\pi(2k+1)/n}$. Only $k = 0$, $z = e^{i\pi/n}$ is inside γ. So we have a simple pole:

$$f(z) = \frac{z - e^{i\pi/n}}{(1+z^n)\left(z - e^{i\pi/n}\right)}.$$

Using the Cover-Up Rule, we get

$$Res_{e^{i\pi/n}} f = \lim_{z \to e^{i\pi/n}} \frac{z - e^{i\pi n}}{z^n + 1} = \lim_{z \to e^{i\pi/n}} \frac{1}{nz^{n-1}} = \frac{1}{ne^{i\pi(n-1)/n}} = \frac{e^{i\pi/n}}{ne^{i\pi}} = -\frac{e^{i\pi/n}}{n},$$

by L'Hospital's Rule. Combining:

$$I = \frac{2\pi i \cdot (-1)^n e^{i\pi/n}}{n(1 - e^{2\pi i/n})} = \frac{\pi}{n} \cdot \frac{-2i}{e^{i\pi/n} - e^{i\pi/n}} = \frac{\pi}{n \sin \pi/n},$$

by Euler's formula. //

We note two special cases:

$$I_2 = \int_0^\infty \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x\right]_0^{\pi/2} = \frac{\pi}{2}; \qquad I_6 = \int_0^\infty \frac{1}{1+x^6} = \frac{\pi}{6 \sin \pi/6} = \frac{\pi}{3}.$$

Second Proof: by the integral of III.6 (keyhole contour). Put $x^n = y$, $x = y^{1/n}$, $dx = (1/n) y^{(1/n)-1} dy$.

$$I = \int_0^\infty \frac{1}{n} . \frac{y^{(1/n)-1} dy}{1+y} = \frac{\pi}{n \sin(\pi/n)}.$$

2. Consider

$$I := \int_0^\infty \frac{x^{1/2}}{1+x^3}\, dx = \frac{\pi}{3}.$$

First Proof: Real Analysis. Put $y := x^{3/2}$, $x = y^{2/3}$. $dx = \frac{2}{3} y^{-1/3}\, dy$. Then

$$I = \int_0^\infty \frac{y^{\frac{1}{3}} \frac{2}{3} y^{-\frac{1}{3}}}{1+y^2}\, dy = \int_0^\infty \frac{2}{3} \frac{1}{1+y}\, dy = \frac{2}{3} \left[\tan^{-1} y\right]_0^\infty = \frac{\pi}{3}.$$

Second Proof: Complex Analysis (we give this for practice in handling many-valued integrands, and in calculating residues). Let $f(z) = z^{1/2}/(1+z^3)$ (double valued, because of the square root – as in Real Analysis!). By ML:

$$\left| \int_{\gamma_2} f \right| = O\left(\frac{R^{1/2}}{R^3} \cdot R \right) \to 0 \qquad (R \to \infty), \qquad \left| \int_{\gamma_4} f \right| = O\left(r^{1/2} \cdot r \right) \to 0 \qquad (r \to 0),$$

$$\int_{\gamma_3} f \to I \qquad (r \to 0,\ r \to \infty).$$

On γ_3, $z = xe^{2\pi i}$, $z^{1/2} = x^{1/2} e^{i\pi} = -x^{1/2}$. So

$$\int_{\gamma_3} f \to \int_\infty^0 \frac{-x^{1/2}}{1+x^3}\, dx = I, \qquad \int_\gamma f \to 2I.$$

By CRT:

$$\int_\gamma f = 2\pi i \sum Res f,$$

and f has singularities where $1 + z^3 = 0$, $z^3 = -1 = e^{i\pi} = e^{(2k+1)i\pi}$, $z = e^{(2k+1)i\pi/3}$, giving (for $k = 0, 1, 2$) $z = e^{i\pi/3}$, $e^{i\pi} = -1$, $e^{5i\pi/3}$. So

$$f(z) = \frac{z^{1/2}}{(z - e^{i\pi/3})(z+1)(z - e^{5i\pi/3})}.$$

Cover-Up Rule:

$$Res_{e^{i\pi/3}} f = \frac{e^{i\pi/6}}{(e^{i\pi/3}+1)(e^{i\pi/3} - e^{5i\pi/3})} = e^{i\pi/6}/[e^{2i\pi/3} - 1 + e^{i\pi/3} - e^{5i\pi/3}]$$

$$= e^{i\pi/6}/[(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) - 1 + (\frac{1}{2} + i\frac{\sqrt{3}}{2}) - (\frac{1}{2} - i\frac{\sqrt{3}}{2})]$$

$$= e^{i\pi/6}/[-\frac{3}{2} + 3.\frac{i\sqrt{3}}{2}] = e^{i\pi/6}/[3.e^{2\pi i/3}] = e^{-i\pi/2}/3 = -i/3.$$

$$Res_{e^{i\pi}} f = \frac{e^{i\pi/2}}{(-1 - e^{i\pi/3})(-1 - e^{5i\pi/3})} = i/[2 + e^{i\pi/3} + e^{5i\pi/3}]$$

$$= i/[2 + (\frac{1}{2} + \frac{i\sqrt{3}}{2}) + (\frac{1}{2} - \frac{i\sqrt{3}}{2})] = i/3.$$

$$Res_{e^{5i\pi/3}} f = \frac{e^{5i\pi/6}}{(e^{5i\pi/3}+1)(e^{5i\pi/3} - e^{i\pi/3})} = e^{5i\pi/6}/[e^{4i\pi/3} - 1 + e^{5i\pi/3} - e^{i\pi/3}]$$

$$= e^{5i\pi/6}/[(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) - 1 + (\frac{1}{2} - i\frac{\sqrt{3}}{2}) - (\frac{1}{2} + i\frac{\sqrt{3}}{2})]$$

$$= e^{5i\pi/6}/[-\frac{3}{2} - 3.\frac{i\sqrt{3}}{2}] = e^{5i\pi/6}/[-3.e^{i\pi/3}] = -e^{i\pi/2}/3 = i/3.$$

So $\sum Res\ f = -i/3$, $2\pi i \sum Res\ f = 2\pi/3$: $2I = 2\pi/3$, $I = \pi/3$. //

To recapitulate on the exam:

Q1: bits and pieces, from anywhere;
Q2,3: theory, Ch. II;
Q4(i),(ii): applications, Ch. III.

As always, I examine on what I have taught (and, usually, emphasized), aiming for a fair test of a working knowledge of the material. I do not expect candidates to leave the exam room feeling surprised.

Use the revision classes next term. Good luck with your revision, and the exam. NHB