m2pm3l5.tex
Lecture 5. 21.1.2010.

Complements.

Draw a Venn diagram with two overlapping sets, showing their intersection and union. We only deal with subsets of a given fixed set, called the *universal set*, Ω (the 'frame in the Venn diagram'). In M2PM3 we take $\Omega = \mathbf{C}$, unless we say otherwise (e.g., $\Omega = \mathbf{C}^*$).
Recall De Morgan's Laws (Augustus De MORGAN (1806-1871), in 1870):

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c : \text{ Complement of union = Intersection of complements;} \\ (A \cap B)^c &= A^c \cup B^c : \text{ Complements of intersection = Union of complements.} \end{aligned}$$

Similarly for arbitrary (e.g. infinite) unions and intersections.

2. Preliminaries from Real Analysis and Topology
Reference: e.g.
W. Rudin: *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976, and
W. Rudin: *Real and Complex Analysis*, 2nd edition, McGraw-Hill, 1974.

1. *Conditional and Absolute Convergence.*
Recall: $\sum_0^\infty a_n$ *converges* means its partial sums $s_n := \sum_0^n a_k$ converge to a limit.
$\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.
Absolute convergence $\Longrightarrow$ convergence; the converse is false (e.g. $\sum_1^\infty (-1)^n/n$ converges, but $\sum_1^\infty 1/n$ diverges).
If $\sum a_n$ converges but not absolutely it is *conditionally convergent.*
Absolutely convergent series behave well under all operation – eg, rearrangement of the order of the terms. Conditionally convergent series do not, and must be handled with care.
2. *Uniform Convergence.*
Defn. A function $f : \mathbf{R} \to \mathbf{R}$ is *continuous* at x_0 if

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ s.t. } \forall x \text{ with } |x - x_0| < \delta, \quad |f(x) - f(x_0)| < \epsilon.$$

Defn. A function $f : [a, b] \to \mathbf{R}$ is *uniformly continuous* on $[a, b]$ if

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ s.t. } \forall x, y \in [a, b] \text{ with } |x - y| < \delta, \quad |f(x) - f(y)| < \epsilon.$$

(That is, if $\delta = \delta(x_0, \epsilon)$ in the Definition above, $\inf_{x_0 \in [a,b]} \delta(x_0, \epsilon) > 0$ - in general, this infimum will be 0).

Defn. $f_n : [a, b] \to \mathbf{R}$ *converges (pointwise)* to $\delta : [a, b] \to \mathbf{R}$ if

$$f_n(x) \to f(x) \quad (n \to \infty) \quad \forall x \in [a, b].$$

Defn. $f_n : [a, b] \to \mathbf{R}$ *converges uniformly* to $f : [a, b] \to \mathbf{R}$ if

$$\forall \epsilon > 0 \, \exists N \text{ s.t. } \forall n \geq N \, \forall x \in [a, b], \quad |f_n(x) - f(x)| < \epsilon,$$

that is,

$$\sup_{[a,b]} |f_n(x) - f(x)| < \epsilon.$$

A series $\sum f_n(x)$ converges (pointwise or uniformly) if its sequence of partial sums $\sum_1^n f_k(x)$ converges (pointwise or uniformly).

We quote:
(i) Weierstrass M-test: if $|f_n(x)| \leq M_n \; \forall x \in [a, b]$ and $\sum M_n < \infty$, then $\sum f_n(x)$ converges uniformly.
(ii) If f_n are continuous, and $f_n \to f$ uniformly, then f is continuous: continuity is preserved under uniform convergence. (This does not hold in general – see Problems 1).

3. *Functions continuous on a closed interval.*
If $f : [a, b] \to \mathbf{R}$ is continuous:
1. f is *bounded*:

$$M = \sup_{[a,b]} f(\cdot) < \infty, \qquad m = \inf_{[a,b]} f(\cdot) > -\infty.$$

2. f *attains its bounds*:

$$\exists x_1, \ x_2 \in [a, b] \text{ with } f(x_1) = M, \quad f(x_2) = m.$$

3. *Intermediate Value Theorem*: f attains every value between its bounds: if $y \in [m, M]$, $\exists x \in [a, b]$ with $f(x) = y$.
4. *Heine's Theorem*: f is *uniformly* continous (on $[a, b]$).
Note. This is *false* if the interval is not closed. E.g., if $f(x) = 1/x$ on $(0, 1]$ (f is *continuous but unbounded* on $(0, 1]$, although f is bounded on $[\epsilon, 1]$ for each $\epsilon > 0$).