m2pm3l6.tex
Lecture 6. 22.1.2010.

4. *Open and Closed Sets; Metric Spaces and Topological Spaces.*

Recall that on $\mathbf{R}$ an interval I with endpoints a, b is *open* if it omits its end-points, $I = (a, b) = \{x : a < x < b\}$, *closed* if it contains its end-points, $I = [a, b] = \{x : a \leq x \leq b\}$. Similarly for rectangles in $\mathbf{R}^2$, cuboids in $\mathbf{R}^3$, etc. In all these cases the definition means that the set is open if the boundaries are omitted, and closed if they are included. It turns out that being an interval, rectangle, cuboid, ... is not the point here. What matters is the difference between openness and closedness. This is very important, can be defined quite generally, and is the basis of General Topology.

Defn. 1. A *neighbourhood* (nhd) of a point x with radius r is $N(x, r) := \{y : |y - x| < r\}$.
2. A *punctured* neighbourhood is $N'(x, r) := N(x, r) \setminus \{x\} = \{y : \ 0 < |y - x| < r\}$.
3. A set S (in $\mathbf{R}^d, \mathbf{C}$,...) is *open* if each point $x \in S$ has a neighourhood in S: $\forall x \in S \, \exists r > 0, \text{ s.t. } N(x, r) \subset S$.
4. A point x of S is a *closure point* of S if each neighbourhood $N(x)$ of x contains a point of S other then x, or, in other words, each punctured neighbourhood $N'(x)$ of x contains a point of S ("meets S").

Example. In $\mathbf{R}$, a, b are closure points of (a, b), but do not belong to it; and they are also closure points of $[a, b]$, and do belong to it.

In $\mathbf{R}^d$, $\mathbf{C}$, but not in general, x is a closure point of $S \iff$ every neighbourhood $N(x)$ meets S in *infinitely many* points of S, i.e. each punctured neighbourhood $N'(x)$ meets S in infinitely many points $\iff$ every neighbourhood $N(x)$ contains an infinite sequence of points in S converges to x (i.e.,x is a *limit point* of S). So in M2PM3 Complex Analysis we may replace 'closure point' by 'limit point', as in the definition below.

Defn. 1. A set S is *open* if every point x in S has a neighbourhood $N(x) \subset S$.
2. A set S is *closed* if S contains all its closure points (equivalently, limit points).

We quote (the proofs are not difficult):

S is open $\Leftrightarrow$ its complement S^c is closed;
S is closed $\Leftrightarrow$ its complement S^c is open.

Defn. The *closure* $\bar{S}$ of S is $\bar{S} := S \cup \{$ closure points of $S\}$.

We quote (the proofs are not difficult):
(i) $\bar{S}$ is closed.
(ii) $\bar{\bar{S}} = \bar{S}$.
(iii) S closed $\Leftrightarrow S = \bar{S}$.
(iv) $\bar{S}$ is the smallest closed set containing S.
(v) $\bar{S}$ is the intersection of all closed sets containing S.

Example. If I is $(a,b), (a,b], [a,b), [a,b]$, then $\bar{I} = [a,b]$.

Defn. $x \in S$ is an *interior point* of S if some neighbourhood $N(x) \subset S$. The set of interior points of S is S^o ("o for open"), the *interior* of S.

The following five statements are the counterparts for open sets of the five above for closed sets. They may be proved in a similar way, or from the above by taking complements.
(i) S^o is open.
(ii) $S^{oo} = S^{oo}$.
(iii) S open $\Leftrightarrow S = S^o$.
(iv) S^o is the largest open subset of S.
(v) S^o is the union of all open subsets of S.

Example. If I is $(a,b), (a,b], [a,b), [a,b]$, then $I^o = (a,b)$.

Defn. The *boundary* ∂S of S is $\partial S = \bar{S} \setminus S^o$.

Example. If I is $(a,b), (a,b], [a,b), [a,b]$, then $\partial I = \{a,b\}$.

Defn. In **C**, write:

$$\begin{array}{lcll} N(z_0, r) & = & \{z : |z - z_o| < r\} & \text{open disc, centre } z_0 \text{ radius } r \\ \bar{N}(z_0, r) & = & \{z : |z - z_o| \leq r\} & \text{closed disc, centre } z_0 \text{ radius } r \\ C(z_0, r) & = & \{z : |z - z_o| = r\} & \text{C for circle.} \end{array}$$

One can check: 1. The union of an *arbitrary* family of open sets is open;

2. The intersection of a *finite* family of open sets is open.

Example. Note that in $\mathbf{R}$, $(-1/n, 1/n)$ is open $(n = 1, 2, 3, \ldots)$, but $\cup_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$, which is closed. So the restriction to 'finite' in 2 is vital.

The two statements above for open sets have 'dual forms' for closed sets, which we obtain by taking complements (which interchanges open and closed) and using De Morgan's laws (which interchange union and intersection):
1. The intersection of an *arbitrary* family of closed sets is closed.
2. The union of a finite family of closed sets is closed.