m2pm3l7.tex

Lecture 7. 25.1.2010.

Note. 1. These properties of open sets form the basis of the subject of General Topology (Felix HAUSDORFF (1868-1942) in 1914), which is about *topological spaces*.
2. More general than $\mathbf{R}^d$ or $\mathbf{C}$ but less general then topological spaces are *metric spaces* (Maurice FRÉCHET (1889-1973) in 1906), spaces with a *distance function* $d = d(x, y)$ between points x, y which satisfy the *Triangle Inequality*

$$d(x, y) \leq d(x, z) + d(z, y).$$

Examples.
1. *Euclidean space* $\mathbf{R}^d$ (including $\mathbf{C}$ as $\mathbf{R}^2$): if $x = (x_1, ..., x_d)$, $y = (y_1, ..., y_d)$,

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{d} |x_i - y_i|^2}$$

(Pythagoras' Theorem: c. 450 BC).
2. *Hilbert Space* (David HILBERT (1862-1943)).
$\ell^2 := \{x = (x_n)_{n_1}^{\infty} : \sqrt{\sum_{n=1}^{\infty} |x_n|^2} < \infty\}$. Then (Pythagoras' Theorem again)

$$d(x, y) = \|x - y\| = \sqrt{\sum_{n=1}^{\infty} |x_n - y_n|^2}.$$

Hilbert space ℓ^2 (a sequence space – there are others, such as the function space L^2 ('L for Lebesgue')) can be thought of as 'Euclidean space of infinite dimension'. There are similarities between Hilbert and Euclidean spaces, but also important differences, which we shall soon meet.

5. *Infinite, countable and uncountable sets.* Write $\mathbf{N}_n = \{1, 2, ..., n\}$.
Defn (Galileo, Dedekind). A set S is *infinite* iff it can be put in 1-1 correspondence with a proper subset of itself, *finite* otherwise.
Example. The simplest infinite set (and the first we meet) is the integers $\mathbf{N}$, which is infinite under this definition because

$$\mathbf{N} \leftrightarrow \mathbf{N} \setminus \{1\} \text{ under } n \leftrightarrow n + 1.$$

We quote: a finite set S can be put in 1-1 correspondence with *exactly one* $\mathbf{N}_n$. This n is called the *cardinality* of S, $card(S)$, or $|S|$.

The Pigeonhole Principle. If there are n staff pigeonholes, and a secretary has n circulars, and has to put one in each: if she ends empty-handed, and knows she hasn't given anyone two, then everyone has one; if she knows she has given everyone (at least) one, then everyone has got exactly one.

This seemingly obvious statement is *Dirichlet's Pigeonhole Principle.* Formally: if a mapping between two finite sets of the same cardinality is surjective, it is also injective, and if injective, it is also surjective. The principle (which may seem at first sight so obvious as to lack content) is in fact extremely powerful. The Pigeonhole Principle can now be seen as essentially equivalent to the Galileo-Dedekind definition of an infinite set: it *holds for finite sets, but fails for infinite sets.*

Defn. If an infinite set S can be put in 1-1 correspondence with $\mathbf{N}$, S is called *countable*, otherwise S is called *uncountable.*

Examples. $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$ are countable, $[0,1], \mathbf{R}, \mathbf{R}^d, \mathbf{C}, \mathbf{C}^d, \ell^2$ are uncountable.

We can think of countable infinite sets as 'small', and of uncountable sets as 'big'. Whereas sums are inherently countable (or finite), integrals – which are limits of sums – are uncountable (in general).

Note. Sums can be regarded as special cases of integrals. In the language of Measure Theory, we use Lebesgue measure for integrals and counting measure for sums.

Complex nth roots of unity. For k integer, $e^{2\pi ik} = 1$. For n integer, take nth roots: $e^{2\pi ik/n} = 1$. These complex values are distinct for $k = 0, 1, \ldots, n-1$, and are called the (*complex*) *nth roots of unity.* They are on the unit circle, equally spaced at the vertices of a regular n-gon (draw a diagram to illustrate this, for $n = 2$, 3, 4, 5 and 6).

If ω is an nth root of unity, it satisfies the equation $\omega^n = 1$. Now $\omega = 1$ is one root. From the identity $\omega^n - 1 = (\omega - 1)(\omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1)$, the other $n-1$ nth roots of unity satisfy

$$\omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1 = 0.$$

If z is complex, $n = 1, 2, 3, \ldots$, and $z^{1/n}$ is one nth root of z, then so are $z^{1/n}\omega_n$, where ω_n runs through the n nth roots of unity. These different values (or branches) are the same when $z = 0$, which is accordingly called a *branch-point* of $z^{1/n}$. There are n nth roots: nth roots are *non-unique.* E.g., for $n = 2$ there are two square roots: even in Real Analysis, we get a sign ambiguity when we take square roots.